2.1 With $S_y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

we know $S_y |+\rangle_y = S_y \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$

$= S_y \frac{\hbar}{2} \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$

so (simpler) also true that $S_y(|+\rangle + i|-\rangle)$

$= \frac{\hbar}{2}(|+\rangle + i|-\rangle)$

So... $S_y \begin{pmatrix} 1 \\ +i \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ +i \end{pmatrix}$ $|+\rangle_x$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ +i \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow$ $a + ib = \frac{\hbar}{2}$ (1)

$c + id = \frac{\hbar}{2} i$ (2)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = S_y \begin{pmatrix} 1 \\ -i \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow$ $a - ib = -\frac{\hbar}{2}$ (3)

$|-\rangle_x$ $c - id = +\frac{\hbar}{2} i$ (4)

$(1)+(3) \Rightarrow 2a = 0 \Rightarrow a = 0$

$(2)+(4) \Rightarrow 2c = \hbar i \Rightarrow c = \frac{\hbar}{2} i$

$(2)-(4) \Rightarrow 2id = 0 \Rightarrow d = 0$

$(1)-(3) \Rightarrow 2ib = \hbar$

$-2i^2 b = -ih$

$2b = -i\hbar$

$b = -\frac{i\hbar}{2}$

$\Rightarrow \hat{S}_y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ +i\frac{\hbar}{2} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$