WORKED EXAMPLES AND CONCEPT EXAMPLE USAGE IN UNDERSTANDING MATHEMATICAL CONCEPTS AND PROOFS

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Elsewhere in this volume, Watson and Mason discuss example generation from the students' perspective by highlighting some of the ways that example generation can be used to increase students' understanding of mathematics and improve their attitudes toward mathematics. This chapter complements this work by describing ways that teachers and textbooks might use examples to help undergraduates better understand mathematics. We distinguish between using worked examples to solve exercises and problems and using examples to help promote students' understanding of mathematical concepts and proofs. We begin with worked examples provided by the teacher or textbook. We then discuss the role of examples in building an understanding of a mathematical concept. Next we discuss how examples can be useful in understanding mathematical proofs. In each of these sections, we present specific suggestions that teachers might use in their own mathematics classrooms and we cite research studies that motivate and support these suggestions.

Worked examples

The term "example" has multiple uses in mathematics education (cf., Watson & Mason, 2002). In some contexts, the word "example" refers to an illustration of a technique used to complete a certain type of mathematical task. For instance, a written solution to the question "Find all local minima and maxima of the function $f(x)=x^3+5x^2-8$" might be regarded as an example of how to solve minimum/maximum problems in an introductory calculus course. This is the way that the word example is often used in undergraduate textbooks, in which individual sections of the book frequently introduce a technique and then provide a series of examples in which the technique is applied. In this paper, we refer to such examples as "worked examples". In other cases, the use of the word example is meant as a particular instance of a mathematical concept (e.g., 6 is an example of an even number). Consideration of these types of examples can be used to improve students' understanding of a concept. Pedagogical uses of examples of concepts will be discussed later in this chapter.

A series of studies by Lithner (2000, 2003, 2004) suggest that undergraduates in procedure-oriented mathematics courses like calculus complete homework exercises predominantly by first locating similar worked examples in the textbook and then using them as a basis for formulating a solution to the exercise. (Note that this might also be the case in proof-oriented courses. Recent studies suggest that some undergraduates in proof-oriented courses may construct proofs via the use of worked examples (Fukawa-Connelly, 2005; Weber, 2004, 2005a, 2005b)). In one study, Lithner (2003) observed undergraduates completing their homework problems in a calculus course. He found that the students in his study almost always used worked examples to complete their homework. This strategy was employed by both weak and strong students, and was used in cases in which the undergraduates had the background knowledge to complete the problems without the use of this strategy (Lithner, 2000, 2003). In another study, Lithner (2004) analyzed the exercises in calculus textbooks to determine what proportion of exercises could be solved by the use of worked examples. He found that, for 90% of the exercises in these textbooks, there was an analogous worked example presented earlier in the section. With minor modifications, these worked examples could be transformed into solutions to the exercises. In these cases, the undergraduates could solve the problem without reasoning about the concepts in the section. Only 10% of the problems could not be solved in this way and required the students to reason about the properties of the concepts that they were ostensibly studying. Lithner (2003, 2004) expressed concern about these results, fearing that many students are completing their homework in ways that are not conducive to building their conceptual understanding or problem-solving strategies.

In spite of Lithner's reservations, many cognitive psychologists have stressed the benefits of having students use worked examples to solve problems (e.g., Zhu & Simon, 1987; Atkinson, Derry, Rankl, & Worthman, 2000). In particular, Atkinson et al. argue that worked examples "provide an expert's problem-solving model for the learner to study and emulate" (p. 181-182). Students who try to solve problems without examples typically develop, practice, and reinforce "novice strategies"—that is, strategies for solving problems that both ignore the deep structure of the problems being solved and are generally ineffective. In contrast, students who use appropriately chosen worked examples as a guide for solving problems are more likely to focus on the deep structure of the problem that they are solving and use more sophisticated strategies for solving it (Atkinson et. al., 2000). Tarmizi and Sweller (1988) analyzed what types of worked examples are most effective for helping students learn how to solve problems. They found that worked examples which reduced students' *cognitive load*—that is, solutions that would not require a student to expend a great deal of mental effort to understand—proved to be more beneficial to students than more complicated worked examples that required greater mental effort to comprehend. For instance, worked examples in geometry that relied on a single mode of representation (e.g., only analytical or only diagrammatic) helped students more than worked examples that combined two representations (e.g., a solution in which analytic reasoning interacted with a diagram), since the latter required students to expend cognitive effort to understand the links between the analytic and diagrammatic portions of the solution.

The preceding summaries serve as a basis for two pedagogical suggestions. First, when worked examples are presented to students, it is important to include examples that are simple and easy to follow. For instance, when presenting a solution to a min/max problem in calculus, it is advisable to have some examples that do not use sophisticated algebraic manipulations, the use of trigonometric identities, or other techniques that an undergraduate might not easily follow. Such examples will cause students to focus more on the details of the solution, rather than its deeper structure. Second, it might be worthwhile to ask students to complete some exercises that cannot be solved solely via the consideration of a worked example, but requires the student to think about relevant properties and concepts (Lithner, 2003). Such experiences will provide students with the opportunity to develop their understanding of the mathematics being studied and mitigate chances that they will develop the unproductive belief that mathematics consists of learning a series of procedures.

Using Examples To Build Concept Images

What does it mean to understand a mathematical concept? An undergraduate's understanding of a mathematical concept should include his or her ability to state and reason from the definition of that concept. However, mathematics educators argue that one's understanding of a mathematical concept involves much more than this. Tall and Vinner (1981) distinguished between a student's knowledge of a concept's definition and that student's *concept image*—i.e., her/his total cognitive structure, including all examples, nonexamples, facts, properties, relationships, diagrams, images, and visualizations, associated with that concept. Students' images of concepts have a significant influence on how they reason about a concept. Many students have images of concepts that are at variance with the concept's definition. For instance, Tall and Vinner (1981) found that many students claimed that functions whose graphs have cusps are not continuous, even if they could state the definition of continuity. Students with poor images of concepts often experience difficulty applying the concept definitions and writing proofs about those concepts (e.g., Moore, 1994; Weber & Alcock, 2004; see also Oehrtman, Selden & Selden, and Harel & Brown, this volume). On a more positive note, students with rich and accurate concept images are often able to reason productively about these concepts and use their intuitive reasoning as a basis for constructing formal proofs (Weber & Alcock, 2004).

These findings suggest that an important goal of mathematics education is to provide students with the opportunity to build strong concept images. In this subsection, we will address the question of how this goal might be achieved. We first present the results of two research studies demonstrating that students can build their understanding of a concept by considering and generating examples of that concept. We then discuss actions that a teacher might take in his or her classroom to lead students to generate or consider a variety of examples and describe research that supports these suggestions.

Several studies show that one way that students can develop a strong concept image is by generating examples of that concept. Dahlberg and Housman (1997) and Housman and Porter (2003) investigated the strategies undergraduate students use to learn a new mathematical concept. In both of these studies, students were given the following formal definition:

A function is called *fine* if it has a root (zero) at each integer.

At first students were given no guidance and were simply asked to come to an understanding of the definition. Students used a variety of learning strategies at this point, including generating examples, reformulating the definition in their own words, memorizing, and recalling definitions of the base concepts *function*, *root*, *at each*, and *integer*. Students were then asked to carry out a number of tasks that both measured and helped to develop their understanding of the new concept: Students were asked to give an example of a fine function, give an example of a function that was not fine (a "nonexample"), provide an explanation in the student's own words and/or pictures of what a fine function is, verify whether six functions were or were not fine, and determine whether four conjectures were true or false. In Dahlberg and Housman's study, the students who used example generation (producing one or more examples related to the concept) and concept reformulation (expressing the concept using pictures, symbols, or words different from the definition) were the ones best able to develop an accurate and useful understanding of the concept. In addition, the students who used example generation were the ones who were best able to identify the correctness of conjectures and provide explanations. The students who primarily reformulated concepts without generating examples were also able to determine whether a given object was an example of the mathematical concept, but these students were more easily convinced of the validity of a false conjecture. Although example generation and concept reformulation were the most beneficial learning strategies for these students, example usage – the use of researcher-provided examples – was also somewhat effective in helping students learn about the concept. In Housman and Porter's study, students who wrote and were convinced by deductive arguments in a separate task-based interview were successful in reformulating concepts, using examples, and generating examples when asked to do so or when it was necessary to disprove a conjecture.

The results of these two studies suggest that the consideration of examples of concepts can help students understand concepts better. In addition, many mathematicians find it useful to consider carefully chosen examples to understand concepts and their definitions (e.g., Alcock, 2004). As Paul Halmos remarked, "A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one." (Halmos, 1983, p. 63). Unfortunately, the data from these two studies also demonstrate that some students do not spontaneously consider examples when presented with a new concept. In the rest of this sub-section, we consider three ways that teachers might lead students to consider examples: (1) by presenting examples, (2) by helping students generate examples, and (3) by asking students to reason about given examples.

The most straightforward suggestion is for teachers to simply provide examples and counterexamples when they introduce a new concept. However, there are two reasons why a teacher should exercise caution in choosing which examples to present. First, students tend to overgeneralize, believing that irrelevant properties held by an example of a concept are shared by all members of the concept. Second, as Watson and Mason point out in this volume, many students will treat counterexamples as isolated cases, anomalies that can be ignored. A few researchers have suggested guidelines for example presentation that can potentially alleviate the negative effects of these student tendencies. We will discuss these guidelines below and then illustrate how they can be applied in the case of convergent sequences.

- First, teachers can present a wide range of examples that do not all share an irrelevant characteristic (Sowder, 1980).
- Second, it is often useful to pair an example and a counterexample that differ in only one characteristic, allowing students to focus their attention on relevant aspects of the concept (Sowder, 1980).
- Third, teachers can not only describe why the examples are, or are not, members of the relevant concept, but also describe ways that students can produce similar examples or counterexamples (Peled & Zaslavsky, 1997).
- Finally, consistent with Watson and Mason's chapter in this volume, after presenting one type of example of a concept, teachers could then ask students to

construct similar examples or even describe how a class of such examples could be constructed (see also Watson & Mason, 2002).

To illustrate these guidelines in a concrete setting, we discuss the concept of convergent sequences. It is natural to exemplify this concept with a prototypical convergent sequence, such as $(1/n)$. The danger with only introducing this one example, or very similar examples such as $(1/n^2)$, is that students may focus on features of this sequence that do not guarantee convergence. For instance, students may infer that convergent sequences are monotonic, never attain their limit, or that each term must be closer to the limit than the last. In fact, an extensive body of research shows that many undergraduates hold these beliefs (Cornu, 1991). For this reason, it is better to present students with a range of examples, perhaps including an alternating sequence converging to zero (illustrating a non-monotonic convergent sequence), a constant sequence (showing that sequences can attain their limits), and non-prototypical convergent sequences such as (1, 2, 3, 4, 5, 6, 1, 1, 1, 1, 1...). Likewise, students should be asked to consider a wide range of counterexamples, including sequences that diverge to infinity and negative infinity, other unbounded sequences, and sequences with multiple cluster points. Each of these counterexamples could be compared to a specific convergent sequence, similar in most respects, but differing in an important respect that causes one to diverge and the other to converge. For instance, comparing the sequences

$a_n = \begin{cases} 1-\frac{1}{n} & \text{if } n \text{ is odd} \\ 1+\frac{1}{n} & \text{if } n \text{ is even} \end{cases}$ and $b_n = \begin{cases} 2-\frac{1}{n} & \text{if } n \text{ is odd} \\ 1+\frac{1}{n} & \text{if } n \text{ is even} \end{cases}$ can illustrate that sequences defined by subsequences can converge, but only if the two subsequences converge to the same number. For examples such as (a_n) and (b_n), teachers can describe the reasoning they used to produce these examples and describe how other examples of that type could be produced. For each presented example, students can be asked to generate another sequence or a family of sequences that have the same features as the example under consideration.

Teachers can also have students play a more active role in their mathematical learning by having them generate examples of concepts themselves. When discussing convergent sequences, the teacher could ask students to generate a *particular* sequence that is convergent. Responses might include $\left(\frac{1}{n}\right)$. Some students may not have come up with any examples on their own, but after seeing their classmates' examples, they might contribute to the next task. The teacher could then ask students to give an example of a sequence that is *peculiar* in some way. Responses might include some of the previously reported examples together with the reasons why they are peculiar. For instance, the sequence $a_n = \begin{cases} 1-\frac{1}{n} & \text{if } n \text{ is odd} \\ 1+\frac{1}{n} & \text{if } n \text{ is even} \end{cases}$ is given as two formulae instead of one, and the sequence $b_n = 0$ for all natural numbers n is constant. Asking students to give examples of convergent sequences in alternative representations, such as a graph of a convergent sequence or expressing a convergent sequence as a recurrence relation, can also be encouraged. Discussion of such examples can lead the class towards a *general* characterization of a convergent sequence (See also Rasmussen & Marrongelle, this volume, for a further description of proactive teacher moves).

Students can also be asked to generate examples in order to explore boundaries and extend the range of possibilities. For example, students who are studying convergent sequences could be asked to find, in this order, each of the following:

1. a convergent sequence,
2. a convergent sequence that is not monotonic,
3. a convergent sequence that does not get strictly closer to its limit with each term,
4. a convergent sequence that achieves its limit, and
5. a convergent sequence whose formula, treated as a real-valued function, would not be continuous.

In doing this task, ask students to make sure that an example given for any one item should *not* satisfy the next item. Thus, the first example would be a sequence that converges, but is monotonic. The second example would be a non-monotonic convergent sequence, but one that does become strictly closer to its limit with each term (such as $\frac{(-1)^n}{n}$), and so forth. When each additional constraint narrows the range of possibilities, students are induced to think more broadly to successfully complete the exercise.

Alcock (2004) suggests a third way teachers can use examples to enrich students' concept images. Students can be given definitions for a collection of concepts. Then students can be presented with worksheets with a number of objects and be asked to determine what properties each object has. To illustrate, after students are introduced to sequences, they can be given the definitions for convergent, bounded, and monotonic sequences. They can then be given a collection of sequences and asked to determine if the sequences are convergent, bounded, and/or monotonic. Conversations between students and between student and teacher can enable students to understand what the definitions of each of these concepts are asserting and to build their concept images of the properties. Further, these activities can address potential misconceptions that students might have or develop (examining sequences such as $\frac{(-1)^n}{n}$ will help students realize that a sequence does not have to be monotonic to converge) and to form mathematical conjectures in response to their own questions (e.g., are all convergent sequences bounded? Do all monotonic bounded sequences converge?). Such a treatment does not have all the benefits of example generation that Watson and Mason discuss in this volume; for instance, the affective benefits of example generation may not be realized here. However, Alcock's suggestion may be more efficient in terms of time, and it does ensure that students will consider the classes of examples that teachers believe are important.

Using Examples In The Form Of Generic Proofs

There are many purposes of presenting proofs in university classrooms. However, mathematics educators argue that two of the most important purposes of proof are *convincing*—i.e., removing all doubt that a theorem is true—and *explaining*—i.e., providing students with insight as to *why* a theorem is true (e.g., Hanna, 1990; Hersh, 1993). Hersh (1993) argues that the formal proofs that we present to our students often fail to achieve both of these goals. First, many students obtain conviction of general assertions not by reading formal proofs, but by checking whether that assertion holds in several individual cases.[1] Formal proofs seem superfluous to these students—they feel they can find out whether an assertion is true or not just by checking a few examples themselves. Further, due both to their weak understanding of formal proofs and the highly formal way that proofs are traditionally presented, undergraduates often do not find proofs to be explanatory (Hersh, 1993).

Rowland (2002) suggests an alternative to formal proofs in number theory. When discussing a general theorem that applies to a class of objects, choose an arbitrary object from among that class. Demonstrate that the theorem holds for that particular object, but make sure that the demonstration relies in no way upon properties of the specific object under consideration that are not shared by all objects in the class to which the theorem applies. Mason and Pimm (1984) call such a demonstration a *generic proof* of the theorem. Rowland advocates presenting generic proofs of theorems prior to, or in lieu of, formal proofs of theorems. He argues that students gain more conviction and understanding from generic proofs than from formal proofs, and students' comprehension of formal proofs will improve if the presentation of a generic proof precedes the presentation of a formal one.

Rowland provides a concrete instance of a generic proof by discussing his treatment of Wilson's theorem. Wilson's theorem asserts:

$(p\text{-}1)! \equiv \text{-}1 \pmod{p}$ for all primes p.

When teaching students about Wilson's theorem, Rowland justifies the theorem using the following generic approach. He first looks at the statement for the particular prime 19, although 13 and 17 would work equally well. He lists the integers between 1 and 18 (inclusively), the reduced set of integers modulo 19. He then draws lines connecting each element in this list to its multiplicative inverse modulo 19, linking 2 with 10, 3 with 13, and so on. Of course, every listed element will have an inverse that is another element in the list, with the exception of 1 and 18, which are their own inverses. Now 18! can be rewritten by lining up each integer with its multiplicative inverse modulo 19. After doing this, Rowland shows how the product $\prod_{i=1}^{18} i \equiv 1 \cdot 1^8 \cdot 18 \pmod{19}$.

There are several aspects of this presentation that made this a good generic proof. The first was Rowland's choice of 19. A prime such as 2, 3, or 5 would not have enough reduced residue classes to see the general structure of Rowland's arguments. A larger prime like 37 would have so many residue classes that the argument would become more difficult to follow; the students may lose the structure of the argument in the arduous calculations of finding the multiplicative inverse of each integer modulo 37. Further, 19 appeared to be an "arbitrary prime"—i.e., it did not have any noticeable distinguishing properties not shared by other primes. Another reason 2 would be a poor prime to inspect was because it was the only even prime.[2] Second, Rowland did not make use of any

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[1] In the language of Harel and Sowder (1998), we might say these students hold an *empirical proof scheme*, but not a *deductive proof scheme*.

[2] The lack of special properties is easier to see if we move beyond looking at primes. For instance, in a generic proof about the natural numbers, one should choose numbers that are neither prime nor perfect squares.

special properties of the number 19 .The central reasoning in Rowland's argument was that every element except 1 and 18 (which is –1 modulo 19) is not its own multiplicative inverse modulo 19. Rowland's demonstration could easily be used to verify Wilson's theorem for any other prime. Third, all constructive aspects of the proof were identified and verified. For instance, the claim that 2 had a multiplicative inverse modulo 19 was not only justified by a theorem. The inverse of 2 was also explicitly found in the number 10, and the student could verify that 2 and 10 were in fact inverses. Finally, the reasoning was presented in such a way that it could easily be abstracted into a more general formal proof. Based on questionnaires and interview data from his own classrooms, Rowland reports that students who see this type of presentation can describe why this general assertion can be applied to any prime number and gain a strong conviction that the theorem holds for all prime numbers.

Following Rowland, Hazzan and Zazkis (2003) describe another way that examples could be used to construct proofs. Many proofs have *constructive components*—they show how certain elements with desired properties can be created, but do not explicitly state what these objects are. Hazzan and Zazkis advocate having students perform the constructions themselves in particular instances before observing the proof that employs these constructions In support of this recommendation, Hazzan and Zazkis argue, "The human mind is not satisfied with the knowledge that some objects exist. There is a desire to point out *exactly* what these objects are. Similarly, unraveling a construction process with an example helps us understand *exactly* how the construction works" (italics are the authors').

Consider the proof that there are infinitely many primes. A standard proof of this proposition is given below.

Theorem: There are infinitely many primes.

Proof (by contradiction): Suppose there are not infinitely many primes. Then we can enumerate the primes $p_1, p_2, \ldots, p_n$. Let $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$. For all i such that $1 \leq i \leq n$, p_i divides $p_1 \cdot p_2 \cdot \ldots \cdot p_n$ but does not divide 1, so p_i does not divide N. Hence, no prime divides N. This contradicts the fact that every integer greater than 1 must be divisible by at least one prime.

This proof has a constructive aspect in that it describes how a number N can be constructed, but does not explicitly state what the number N is. The exact value of N, of course, depends on what numbers are elements of the hypothetical finite set of primes (Leron, 1985). Students often have trouble following this proof. However, if students were asked to construct the N in the proof for particular sets of primes, their understanding might improve. For instance, students could verify that:

2 and 3 do not divide $N = 2 \cdot 3 + 1$

2, 3, and 5 do not divide $N = 2 \cdot 3 \cdot 5 + 1$.

2, 3, 5, and 7 do not divide $N = 2 \cdot 3 \cdot 5 \cdot 7 + 1$, and so on.

Students might also want to examine $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$. Here 30031 is a composite number ($30031 = 59 \cdot 509$). Inspecting this example can make students aware that the product of the first n primes plus one does not always yield a prime number, only a number whose prime factors are not included in the presumably finite set of primes.

After exploring these examples, the formal proof that there are infinitely many primes will be more accessible to students. They will have a greater appreciation for how the variable N in the proof is being defined and why none of the enumerated primes will divide it.

Hazzan and Zazkis further illustrate how these techniques can be used to enhance students learning of other proofs with constructive components. One proof they looked at was a standard proof of the Basis Theorem in linear algebra. The Basis Theorem asserts that, in a finite dimensional space, all bases have the same cardinality. Standard proofs of the Basis Theorem often rely on the Replacement Lemma, which asserts: "Let B be a set of linearly independent vectors in the spanning space of a set of vectors A. For all subsets $B_1 \subseteq B$, there exists a subset $A_1 \subseteq A$, such that $|A_1| = |B_1|$ and $(A\text{-}A_1) \cup B_1$ spans the same space as A."

The Replacement Lemma is clearly constructive, in the sense that it tells the reader that a subset of spanning set exists, but it does not state what it is or even how it could be found. As a result, many students find proofs of the Basis Theorem relying on this lemma to be confusing. Hazzan and Zazkis (2003) designed a series of computer activities to help students understand the Replacement Lemma. These activities allowed students to enter a spanning set A and a set of linearly independent elements B_1 and the computer would then find the elements in the spanning set which could be replaced by the subset B_1. Students who completed these exercises found the subsequent proof of the Basis Theorem to be understandable and meaningful.

Conclusion

In this chapter, we have discussed a number of ways that teachers can use worked examples and employ examples to build undergraduates' understanding of mathematical concepts and proofs. Examples not only illustrate concepts, principles, and proofs, they can help students to explore, expand, generalize, refine, and test their understanding. Students who are exposed to, work with, and generate their own examples are actively engaged in mathematics and learning.

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Responses to Review 1

Suggested Change	Action Taken
Add a short discussion of the possible pitfalls of using examples if the guidelines suggested here are ignored.	We believe that we had discussed possible pitfalls of ignoring the suggested guidelines. In our revision, we added sentences to make the potential pitfalls more prominent and clear. For instance, we added sentences such as the following: For instance, when presenting a solution to a min/max problem in calculus, it is advisable to have some examples that do not use sophisticated algebraic manipulations, the use of trigonometric identities, or other techniques that an undergraduate might not easily follow. Such examples will cause students to focus more on the details of the solution, rather than its deeper structure.
If the authors have examples of student work, providing these could enhance the usefulness, clarity, and convincingness of the paper.	We did not have examples of students' work that we could use that would not considerably lengthen the paper and distract from its focus.

Responses to Review 2

Suggested Change	Action Taken
p. 310: Omit the first sentence.	We did this.
p. 310: Begin the third sentence this way: *This chapter complements their work by describing...*	We did this.
p. 310: Final sentence of the first paragraph (and elsewhere): *mathematics* classrooms; not *mathematical* classrooms.	We made these changes throughout the paper.
p. 310: Paragraph 2, line 1: Omit *use of the term*	We did this.
p. 310: Paragraph 2, line 10: Replace *is meant as an example* with *is meant as a particular instance.*	We did this.
p. 310: Paragraph 3, line 2: Replace *procedural-oriented* with *procedure-oriented..*	We did this.
p. 310: Paragraph 3, lines 4-7: You have here a confusing plethora of parentheses.	We moved the parenthetical reference citation to the end of this sentence to improve the sentence's clarity.
p. 311: Line 8: Instead of *session* do you mean *section*?	Yes. We changed this word.
p. 311: Paragraph 2, lines 6-7: Consider omitting *or shallow strategies for solving problems.*	We omitted this wording.
p. 314: 3rd, 4th, and 5th lines from bottom: Is	We adjusted the punctuation to clarify the

there one statement following the colon, or two? If one, what do you mean? If two, re-punctuate for clarity.	meaning.
p. 315: In the paragraph beginning "Alcock (2004)", line 12: It would be helpful to add as follows: …and to form mathematical conjectures *in response to their own questions (e.g.,…).*	We made this addition.
p. 316: Line 7: …find proofs *to* be…	We added the word 'to', as suggested.
p. 316: Paragraph 3, line 3: Remove italics from -1.	We did this.
p. 316: Paragraph 4, line 8: Every listed element has an inverse; 1 and 18 are not exceptions. Revise your language to say what you really meant, that each element of the list except for 1 and 18 can thus be paired with another element in the list.	We revised the language, as suggested.
p. 316: Paragraph 5, line 7: Consider adding the word *noticeable* to get *noticeable distinguishing properties...* It is pretty hard to find a number that doesn't have **any** distinguishing properties…	We added this word.
p. 317: Line 3: …could ~~be~~ easily be…	We omitted (the first occurrence of) the word 'be', as suggested.
p. 317: In your proof that there are infinitely many primes, you draw an incorrect conclusion. It does not follow that N must be prime (as you actually note elsewhere).	We corrected this proof, as suggested.
p. 318: Line 10: …*linearly* independent vectors…	We added this word.
p. 317: Line 11: …there exists a subset ~~of~~ $A_1 \subseteq A$	We deleted the word 'of'.

Responses to Review 4

Suggested Change	Action Taken
No changes suggested.	none