Modeling a Biological Auction

Valparaiso University Talk

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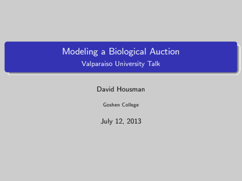
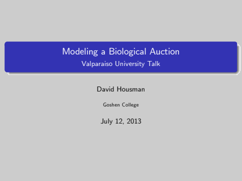
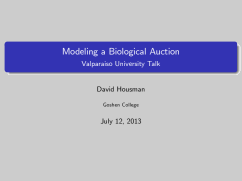
July 12, 2013



Two animals vie for a prize: territory, food, or a mate. They display their prowess and may even fight. Eventually a winner emerges and the loser sulks away. This is analogous to an auction where the loser, as well as the winner, must pay his bid. A mathematical model of this scenario and its solution involves concepts and techniques from undergraduate probability and calculus: probability density, mathematical expectation, function maximization, fundamental theorem of calculus, inverse functions, and the derivatives of inverse functions. A complete derivation and surprising conclusions are given.



The audience will participate in an auction with a somewhat unusual rule. We will explain why this auction models certain biological phenomena. Finally, techniques from calculus will be used to determine the biologically optimal strategy.



Game theory is about the interaction among people, nations, animals, genes, or other agents. In their attempt to understand, recommend, or predict behavior in situations of potential conflict and cooperation, game theorists develop and analyze mathematical models. Experimentalists have added empirical critiques and enhancements to the models. In order to illustrate some of the game theory and experiments, the audience will participate in a few games (perhaps winning some money!). Applications to economics and biology will be discussed.

Outline

- A strange auction
- The biological connection
- A strange auction repeated
- Best response to a known opponent
- Biologically optimal strategy
- Evolution of strategies
- Concluding remarks

A Strange Auction

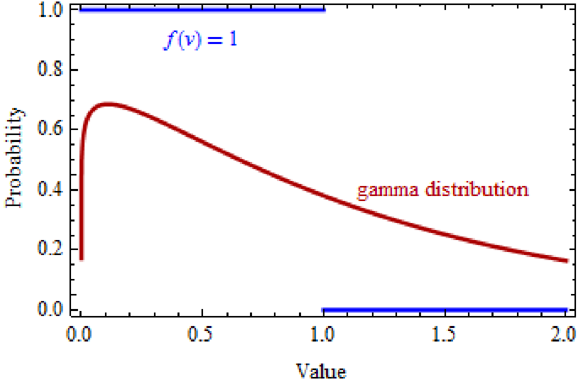
- Open ascending bid auction for a prize.
- The highest bidder wins the prize and pays her bid.
- The second highest bidder wins nothing but pays his bid.
- No one else pays.
- Play now!
- Biological interpretation.

A Strange Auction Repeated

- The value of the prize to you is on the paper and was drawn from a uniform distribution on 0 to 1000.
- Sealed (nonnegative) bid auction for the prize.
- Both pay the lower bid, but only the higher bidder wins the prize.
- If there is a tie, both pay their bid and each wins $1/2$ of the prize.
- Repeat 30 times with a variety of opponents.
- Keep track of the strategy you use and its effectiveness.
- Play now!
- What were the most effective strategies?

Strange Auction Model I

- Both players pay the lower bid, but only the higher bidder wins the prize.
- A player knows what the prize is worth to him/her but not what it is worth to his/her opponent.
- $f(v)$ is the probability density the prize is worth v to a player.



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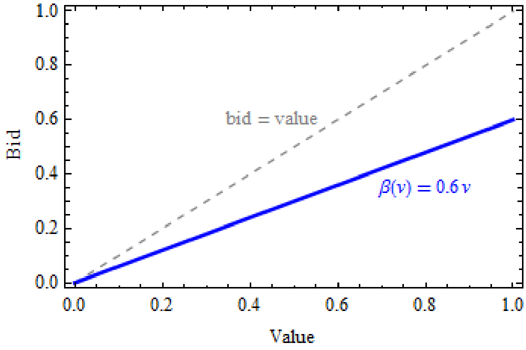
- $\beta(v)$ is the opponent's bid if the prize is worth v to him.



Strange Auction Model I

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- A player knows what the prize is worth to him/her but not what it is worth to his/her opponent.
- $f(v)$ is the probability density the prize is worth v to a player.
- $\beta(v)$ is the opponent's bid if the prize is worth v to him.
- If I value the prize at v and bid b, my expected payoff is

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u)) f(u)\, du - b \int_{\beta(u)\geq b} f(u)\, du$$

- I want to choose $b \geq 0$ to maximize $\pi_v(b)$.

Payoff Maximization (General Case)

- Maximize the following at $b = b^*$:

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u)) f(u)\, du - b \int_{\beta(u)\geq b} f(u)\, du$$

- Assume β is strictly increasing and F is the cdf of f.

$$\pi(b) = \int_0^{\beta^{-1}(b)} (v - \beta(u)) f(u)\, du - b(1 - F(\beta^{-1}(b)))$$

- Assume β is differentiable.

$$\pi'(b) = \frac{(v - \beta(\beta^{-1}(b))) f(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))} - (1 - F(\beta^{-1}(b))) + \frac{bf(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}$$

- Simplify.

$$\pi'(b) = vf(\beta^{-1}(b))/\beta'(\beta^{-1}(b)) - (1 - F(\beta^{-1}(b)))$$

- First order necessary condition $\pi'(b^*) = 0$.

$$0 = vf(\beta^{-1}(b^*))/\beta'(\beta^{-1}(b^*)) - (1 - F(\beta^{-1}(b^*)))$$

Payoff Maximization (Special Case)

- First order necessary condition.

$$0 = \pi'(b^*) = vf(\beta^{-1}(b^*))/\beta'(\beta^{-1}(b^*))) - (1 - F(\beta^{-1}(b^*)))$$

- Suppose $f(v) = 1, v \in [0,1]$ and $\beta(v) = av, v \in [0,1]$. Hence, $F(v) = v, v \in [0,1]$ and $\beta^{-1}(b) = b/a, b \in [0,a]$.

$$0 = v \cdot 1/a - (1 - b^*/a)$$

- Solve for b^*.

$$b^* = a - v$$

- We have found a local minimum!

$$\begin{aligned} \pi'(b) &= v/a - 1 + b/a \\ \pi(b) &= (v/a - 1)b + (1/2a)b^2 \end{aligned}$$

- The correct maximum is a trigger strategy.

$$b^* = \begin{cases} 0, & \text{if } v \le a/2 \\ a, & \text{if } v \ge a/2 \end{cases}$$

Strange Auction Model II

- Both players pay the lower bid, but only the higher bidder wins the prize.
- A player knows what the prize is worth to him/her but not what it is worth to his/her opponent.
- $f(v)$ is the probability density the prize is worth v to a player.
- $\beta(v)$ is the opponent's bid if the prize is worth v to him.
- If I value the prize at v and bid b, my expected payoff is

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u)) f(u)\, du - b \int_{\beta(u)\geq b} f(u)\, du$$

- Assume $\beta(v)$ is the player's payoff maximizing bid, that is,

$$\pi(\beta(v)) \geq \pi(b)$$

for all $b \geq 0$.

Payoff Maximization (General Case)

- Maximize the following at $b = \beta(v)$:

$$\pi(b) = \int_{\beta(u)<b} (v - \beta(u)) f(u)\, du - b \int_{\beta(u)\geq b} f(u)\, du$$

- As before, take the derivative.

$$\pi'(b) = vf(\beta^{-1}(b))/\beta'(\beta^{-1}(b))) - (1 - F(\beta^{-1}(b)))$$

- First order necessary condition $\pi'(\beta(v)) = 0$.

$$0 = vf(v)/\beta'(v) - (1 - F(v))$$

- Solve for β'.

$$\beta'(v) = \frac{vf(v)}{1 - F(v)}$$

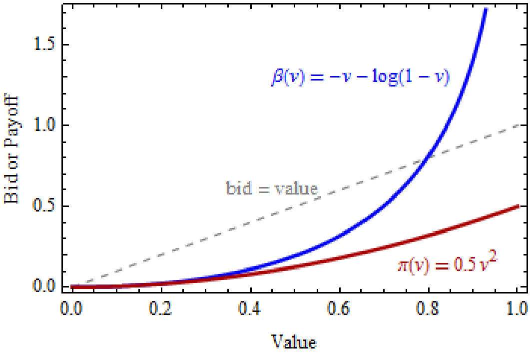
- Solve for β.

$$\beta(v) = \int_0^v \frac{uf(u)}{1 - F(u)}\, du$$

- This function is differentiable and increasing from $\beta(0) = 0$.

Payoff Maximization (Special Case)

- Suppose $f(u) = 1, u \in [0,1]$ and $F(u) = u, u \in [0,1]$.
- $\beta(v) = \int_0^v \frac{uf(u)}{1-F(u)}\,du = \int_0^v \frac{u}{1-u}\,du = -v - \ln(1-v)$.
- $\pi_{\max}(v) = \frac{1}{2}v^2$.



- Find the probability that the prize is won and too much is paid.
- Repeat the analysis if only the winner pays the lower (or higher) bid.

Payoff Maximization Verification

- To verify we have found a maximum, substitute

$$\beta'(v) = \frac{vf(v)}{1 - F(v)}$$

- into

$$\pi'(b) = vf(\beta^{-1}(b))/\beta'(\beta^{-1}(b))) - (1 - F(\beta^{-1}(b)))$$

- to obtain

$$\pi'(b) = (1 - F(\beta^{-1}(b))(v/\beta^{-1}(b) - 1)$$

- which is positive if $b < \beta(v)$
- and negative if $b > \beta(v)$.

Payoff Using the Strategy

- The payoff to a player who values the prize at v and bids b

$$\pi(b) = \int_0^{\beta^{-1}(b)} (v - \beta(u))f(u)\,du - b(1 - F(\beta^{-1}(b)))$$

- is maximized at $b = \beta(v)$

$$\pi_{\max}(v) = \int_0^{v} (v - \beta(u))f(u)\,du - \beta(v)(1 - F(v))$$

- Hence,

$$\pi_{\max}(0) = 0$$

- Taking the derivative

$$\begin{aligned}
\pi'_{\max}(v) &= (v - \beta(v))f(v) + F(v) - \beta'(v)(1 - F(v)) + \beta(v)f(v) \\
&= vf(v) + F(v) - \frac{vf(v)}{1 - F(v)}(1 - F(v)) \\
&= F(v) \geq 0
\end{aligned}$$

- The more you value the prize, the higher your expected payoff.

Surprising Observation

- Recall the optimal bidding strategy.

$$\beta(v) = \int_0^v \frac{uf(u)}{1-F(u)}\,du$$

- Find the average bid.

$$\int_0^\infty \beta(v)f(v)\,dv = \int_0^\infty \int_0^v \frac{uf(u)}{1-F(u)}du\ f(v)\,dv$$

- Interchange integrals ($0 \le u \le v < \infty$).

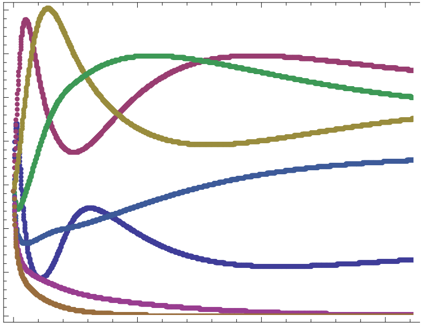
$$\int_0^\infty \beta(v)f(v)\,dv = \int_0^\infty \frac{uf(u)}{1-F(u)} \int_u^\infty f(v)\,dv\,du$$

- Since the inner integral is $1 - F(u)$,

$$\int_0^\infty \beta(v)f(v)\,dv = \int_0^\infty uf(u)\,du$$

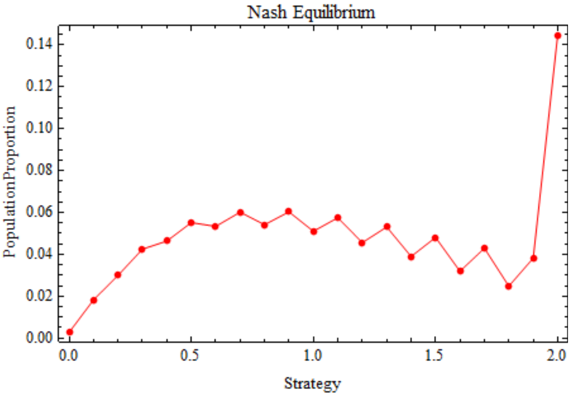
- The average bid equals the average value.
- For some prize values v, the bid $\beta(v)$ is greater than the value!

Evolution with Fixed Set of Proportional Strategies



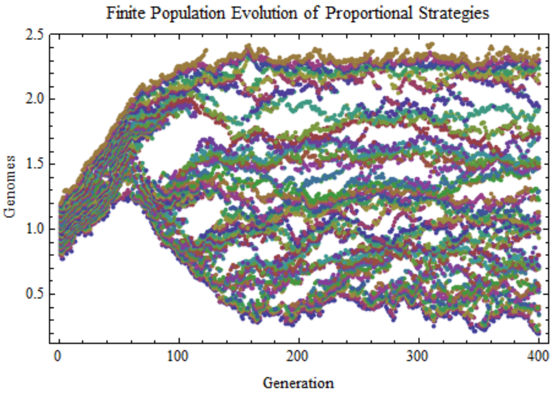
- The stable distribution is $p(0) = 5/65$, $p(.5) = 17/65$, $p(1) = 16/65$, $p(1.5) = 15/65$, $p(2) = 12/65$, $p(2.5) = 0$, $p(3) = 0$.
- Lot's of strategies could invade.

Evolution with Fixed Set of Proportional Strategies



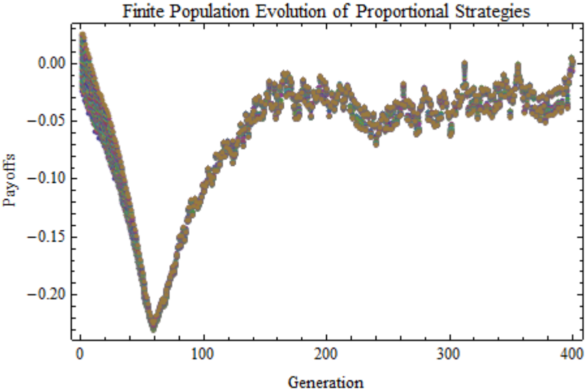
- Why is this distribution stable?

Evolution with Fixed Set of Organisms



- Start with 0.80, 0.81, 0.82, ..., 1.19, 1.20. Remove the poorest performer, duplicate the best performer, and mutate with normal distribution with standard deviation 0.02.

Evolution with Fixed Set of Organisms



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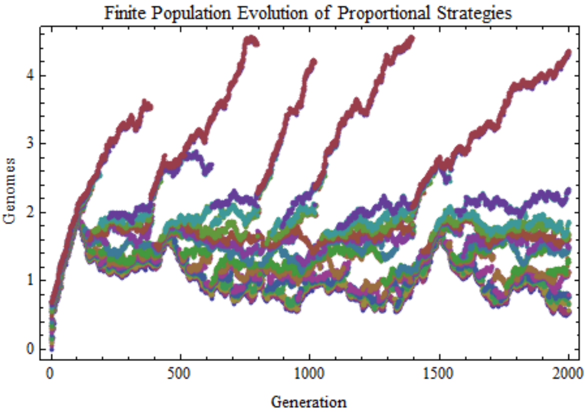
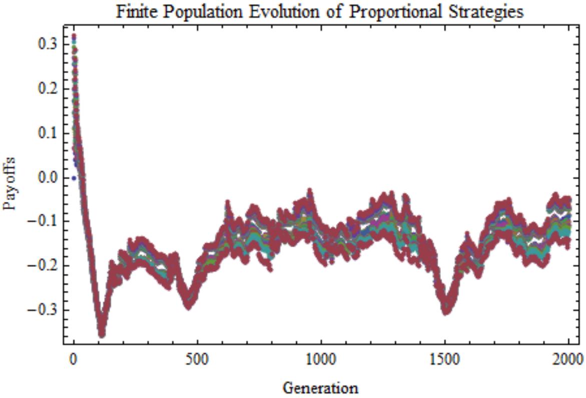


- Start with 0.00, 0.05, 0.10, ..., 0.65, 0.70. Remove the poorest performer, duplicate the best performer, and mutate with normal distribution with standard deviation 0.02.

Evolution with Fixed Set of Organisms



- Start with 0.00, 0.05, 0.10, ..., 0.65, 0.70. Remove the poorest performer, duplicate the best performer, and mutate with normal distribution with standard deviation 0.02.

Conclusions

- Interesting biological phenomena can be modeled with mathematics.
- Different models, based on different assumptions, can result in different conclusions.
- Interesting questions remain unanswered.
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- Questions?