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REAPPORTIONMENT BY WEIGHTED VOTING

by

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PREFACE

The decennial census gives rise to several problems involving mathematics and fair political representation. The constitutional reason for the census every ten years is to apportion the seats (currently fixed at 435) of the U.S. House of Representatives among the 50 states according to the populations of the states. A superb exposition of the apportionment problem in U.S. history is presented in the forthcoming book Fair Representation by M.L. Balinski and H.P. Young, Yale University Press, 1982. Shifts in population figures and changes in the number of representatives also brings about the need for redistricting or other forms of reapportionment at the state and local levels of government.

The principle of "one person, one vote" is normally enforced at the national and state level so as to require legislative districts of approximately equal size. The courts have also ruled that this principle applies at the local level. However, reapportionment at the local level can often be achieved by means of either weighted voting or multimember districts, as well as by redistricting. The former methods may prove desirable in order to have representatives' (or voting) districts correspond to preexisting municipalities of unequal population sizes.

Weighted voting systems have become fairly common in local governing bodies in New York State. About half of the county boards of representatives in the State have implemented or at least seriously considered weighted voting since the late 1960s. The courts in New York State have accepted the Banzhaf power index (also called the Coleman value or Chow parameters) as an appropriate measure for such weighted voting systems. The goal is to weigh the representatives' votes so that the computed Banzhaf indices are nearly proportional to the corresponding populations. This calculation normally requires the aid of a computer or good programmable calculator. Weighted voting is thus frequently referred to as "computerized voting." Many counties will consequently hire a specialized consultant to undertake such computations. However, since the 1980 census, some counties (e.g., Cortland and Tompkins) have instead asked local mathematicians or computer scientists to determine a suitable set of weights, and at about 25% of the cost of a consultant.

The following study was undertaken in order to introduce a weighted voting system in Tompkins County, New York, in which the authors reside.

Multimember districts are not allowed in this case. So the County Board had to redistrict or introduce weighted voting. The Republican members of the board who were in an 8 to 7 majority (currently it is 9 to 6) did not want to redistrict and "cut up" parts of existing townships (or city wards), and they were willing to support a weighted system to avoid this. The Democratic Party went to court in an attempt to achieve redistricting, but the judge approved the weighted voting plan as a temporary solution until 1985. (Many other rulings in the State have not been only temporary.) Furthermore, the judge ruled (in late December, 1981) that this solution could not be taken to the people in a referendum, because it was only a "temporary" solution.

The Tompkins County Board of Representatives has 15 members: 5 from separate districts in the City of Ithaca, 3 from districts in the Town of Ithaca, 2 from districts in the Town of Dryden, and one each from the Towns of Groton, Lansing, Ulysses, Enfield & Newfield, and Caroline & Danby. Appendices I and II of this report give seven possible sets of weights for each of the simple and the two-thirds majority cases. Tables I.7 and II.7 were the cases we recommended for the respective majorities. After some discussion the Board selected to use Table I.7 and II.2. Table II.2 was our second choice for the two-thirds majority, but the Board did not know that in advance. (Note that the final zero in the weight column C in Table II.2 can be deleted.) The judge then approved these selections and the weighted voting plan was introduced starting in January 1982.

Tompkins County also came up with several possible plans for redistricting into more equal sized districts, and will likely need to undertake redistricting in a few years. There also exists several mathematical approaches to this problem in the literature. It should be noted that some counties (e.g., Cortland) redistrict to within 5% and then do weighted voting as well.

WEIGHTED VOTING IN TOMPKINS COUNTY

1. Introduction. Reapportionment for county boards in New York State has been achieved by means of the following three methods.

(1) Redistricting the representative regions by altering their boundaries so that each such election district has nearly equal population.

(2) Weighted voting (also called "computerized" voting) in which each representative's vote is weighted in such a manner that his or her resulting power relates to the number of people represented.

(3) Multimember districts in which more than one representative may be elected at-large from the same district. A variant of this is the floterial case in which some members are elected from smaller districts whereas others may be elected at large from larger regions that cover or overlap some of the smaller districts.

This report is concerned with weighted voting in Tompkins County, New York in the early 1980s using the existing election districts and the 1980 U.S. census data. Some preliminary remarks about weighted voting, measuring power, and weighted voting in counties in New York State appear in sections 2,3 and 4. Some discussion about good solutions and computational aspects appears in sections 5 and 6. Several possible weighted voting systems for Tompkins County for the case of a simple majority of the board and for the case of a two-thirds majority are given in the tables appearing in appendices I and II, respectively. Explanations for these tables and recommendations appear in sections 6 and 7. Some additional documentation and computer programs appear in appendix III.

2. Weighted Voting. Weighted voting is frequently used when there is sufficient reason to create or maintain districts which have nontrivial variations in populations. This may be caused, e.g., by a desire to maintain some correspondence between election districts and existing municipalities or natural geographical boundaries. More than one-third of the counties in New York State have made use of such weighted voting systems. The object of such systems is to create equity by weighing the representatives' votes so as to properly adjust for the fact that they

represent different numbers of constituents.

In a weighted voting system there are n voters (representatives) denoted by

$$1, 2, 3, \ldots, n;$$

each voter has a corresponding weight (positive number)

$$w_1, w_2, w_3, \ldots, w_n;$$

and there is a quota q necessary to pass an issue. The number q may be a simple majority, a two-third majority, etc. A coalition (subset) of voters can pass a resolution if and only if their total weight is at least q; i.e., if the sum of the weights of the voters in this coaltion adds to q or greater. Such a coalition is called winning. All other coalitions are losing. Note that the role of the weights w_1, w_2, ..., w_n and quota q is to determine which coalitions of voters are winning.

One common and rather intuitive way to attempt to correct for districts of unequal population is to weigh the representatives' votes so that they are directly proportional to the number of people they represent. This approach frequently produces good results in terms of the relative powers of the representatives. However, there can arise cases in which the relative weight of one's vote is not a good measure of one's power. E.g., if four voters 1, 2, 3 and 4 had the weights $w_1 = 43$, $w_2 = 42$, $w_3 = 9$ and $w_4 = 6$ with a simple majority of $q = 51$ necessary to win, then the first three voters have equal power in affecting decisions whereas voter 4 has no power at all. So it is not always sufficient to weigh representatives' votes directly proportional to their respective populations.

Instead, one must introduce an intermediate step. One should assign weights to voters so that their resulting "power" to influence decisions is proportional to the number of people being represented. In New York State the courts accept a measure for power introduced by John F. Banzhaf III, a lawyer currently at Georgetown University, as an appropriate measure to use in such weighted voting situations. Therefore, the goal is to assign representatives weights so that their resulting Banzhaf power indices are directly proportional to their respective

populations.

3. Power Indices. In general the notion of power is a difficult concept to quantify. However, the ability to pass resolutions in a given weighted voting system does exhibit some quantitative structure and properties, and several measures for such power have been proposed. The first serious measure of this type was the Shapley-Shubik index (or Shapley value (1953)). The one with legal precedence in the courts is the Banzhaf index (also called the Banzhaf-Coleman value). For many voting situations these two measures give quite similar results, but this is not always the case.

The Banzhaf power index is defined as follows. Consider all theoretically possible ways that the group of n voters can be partitioned into two coalitions, one in favor of an issue and the other against it. I.e., all possible combinations of the n voters into "yea" and "nay" groups. The number of such distinct combinations is 2^n, i.e., the same as the number of subsets of n elements. A given voter is decisive (or is called a swing) in a particular combination if by changing his or her vote (from yea to nay), while all other votes remain unchanged, the result of the group's vote will be reversed (from pass to fail). One also refers to such a voter in this particular combination as pivotal, critical or marginal. The Banzhaf index merely counts for each voter the number of combinations in which he or she is a swing. For any two voters it is only the relative magnitude of their number of swings which is important in comparing their power. So one can multiply such indices by a common constant. It is thus natural to express the Banzhaf index in a normalized form in which the sum of the indices for the n voters is a convenient constant such as 1 or n. (In the tables in this report the Banzhaf indices are normalized so that they sum to $n = 15$.)

The Banzhaf index has a probabilistic interpretation. If each combination of "yea" and "nay" votes is equally likely to occur, then a voter's index is proportional to the probability he will be decisive in a given vote. The Banzhaf index also has a more theoretical definition in terms of axioms. The more mathematically mature reader can consult the paper by Dubey and Shapley given in the references. Elementary

explanations of the Banzhaf index along with several illustrations appear in the three papers by him, and in the one paper by Johnson, listed in the references.

4. New York State. The federal courts have ruled that the "one person, one vote" concept applies to local governments, school district trustees, and political party structures. Some brief highlights of early court dicisions in New York State regarding weighted voting and "one person, one vote" follows.

1966 In 1967, the Court of Appeals of New York ruled (Graham v. Board of Supervisiors of Erie County) that weighted voting may be approved "solely as a temporary expedient; but that a permanent plan must be based on the principle of 'one man, one vote'". At that time weighted voting could only be employed as a stopgap measure, to be used while a new apportionment system was being instituted. A year later, in a landmark case (Town of Greenburgh v. Board of Supervisors) the New York Court of Appeals upheld the constitutionality of a weighted voting system based on population. By the mid 1970s, at least twenty-two of the fifty-seven counties of New York (outside of New York City) had adopted weighted voting schemes for their county board of legislators. However to conform with the ruling of the Court of Appeals ruling in Iannucci v. Board of Supervisors (1967), a "computerized analysis" must be presented to validify the Apportionment plan.

Appellate Division in 1965

> "It is improper for a court in passing upon a constitutional question, to lightly disregard the considered judgement of a legislative body which is also charged with duty to uphold the Constitution but with respect to weighted voting a considered judgement is impossible without computer analysis and, accordingly if county board of supervisors chose to reapportion themselves by use of weighted voting there is no alternative but to require them to come forward with such analysis and demonstrate the validity of their apportionment plan."

A detailed discussion of the early history of weighted voting in county boards in New York State is given in the 1969 reference by Johnson.

Many of the counties in New York State have employed the consultant Lee Papayanopoulos, a professor in the School of Business at Rutgers University, to do the necessary computer calculations for their

weighted voting plans. However, the mathematical or computer scientist so inclined can perform or check such calculations if they have an appropriate computer or calculator available.

Many examples of weighted voting in several other contexts appear in the references by Lucas and by Barrett and Newcombe.

5. Calculations and Approximations. The objective of a weighted voting system is to have the voters' (i.e., representatives') powers proportional to the populations they represent. The goal of this report is to determine what weights should be selected for the representatives so that their resulting Banzhaf indices will be proportional to the respective populations for the election districts in Tompkins County.

Two problems of a computational nature arise. First, not all n-tuples (of positive numbers that sum to 15) can arise as a (normalized) Banzhaf index. So no theoretically perfect answer may be attainable for a given set of population data. One must resort to good approximations to an ideal solution, i.e., weights for which the resulting Banzhaf indices are close to being proportional to respective populations. Second, there is no computationally feasible way to solve for the "best" set of weights, given the desired Banzhaf indices. So one must resort to doing the problem in the opposite order. One first "guesses" at a set of reasonable weights and then computes the corresponding indices, and examines the latter to see if they give a good approximation. Solving for indices, given the weights, is a feasible (though long) calculation. This latter process of selecting weights and computing indices normally has to be repeated many times, until one arrives at weights for which the indices are good approximations to the desired powers.

Weighted voting has frequently been referred to as "computerized voting" because the necessary calculations (for all but small values of n) requires a computer or programmable calculator. For 15 voters one must consider $2^{15} = 32{,}768$ combinations, and each voter will be decisive in a few thousand such combinations. So for each set of weights (and quota q) selected, one must perform many thousands of (simple) arithmetical operations to determine the Banzhaf indices. And this process is normally repeated many times in order to test many potential