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REAPPORTIONMENT BY WEIGHTED VOTING

by

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PREFACE

The decennial census gives rise to several problems involving mathematics and fair political representation. The constitutional reason for the census every ten years is to apportion the seats (currently fixed at 435) of the U.S. House of Representatives among the 50 states according to the populations of the states. A superb exposition of the apportionment problem in U.S. history is presented in the forthcoming book Fair Representation by M.L. Balinski and H.P. Young, Yale University Press, 1982. Shifts in population figures and changes in the number of representatives also brings about the need for redistricting or other forms of reapportionment at the state and local levels of government.

The principle of "one person, one vote" is normally enforced at the national and state level so as to require legislative districts of approximately equal size. The courts have also ruled that this principle applies at the local level. However, reapportionment at the local level can often be achieved by means of either weighted voting or multimember districts, as well as by redistricting. The former methods may prove desirable in order to have representatives' (or voting) districts correspond to preexisting municipalities of unequal population sizes.

Weighted voting systems have become fairly common in local governing bodies in New York State. About half of the county boards of representatives in the State have implemented or at least seriously considered weighted voting since the late 1960s. The courts in New York State have accepted the Banzhaf power index (also called the Coleman value or Chow parameters) as an appropriate measure for such weighted voting systems. The goal is to weigh the representatives' votes so that the computed Banzhaf indices are nearly proportional to the corresponding populations. This calculation normally requires the aid of a computer or good programmable calculator. Weighted voting is thus frequently referred to as "computerized voting." Many counties will consequently hire a specialized consultant to undertake such computations. However, since the 1980 census, some counties (e.g., Cortland and Tompkins) have instead asked local mathematicians or computer scientists to determine a suitable set of weights, and at about 25% of the cost of a consultant.

The following study was undertaken in order to introduce a weighted voting system in Tompkins County, New York, in which the authors reside.

Multimember districts are not allowed in this case. So the County Board had to redistrict or introduce weighted voting. The Republican members of the board who were in an 8 to 7 majority (currently it is 9 to 6) did not want to redistrict and "cut up" parts of existing townships (or city wards), and they were willing to support a weighted system to avoid this. The Democratic Party went to court in an attempt to achieve redistricting, but the judge approved the weighted voting plan as a temporary solution until 1985. (Many other rulings in the State have not been only temporary.) Furthermore, the judge ruled (in late December, 1981) that this solution could not be taken to the people in a referendum, because it was only a "temporary" solution.

The Tompkins County Board of Representatives has 15 members: 5 from separate districts in the City of Ithaca, 3 from districts in the Town of Ithaca, 2 from districts in the Town of Dryden, and one each from the Towns of Groton, Lansing, Ulysses, Enfield & Newfield, and Caroline & Danby. Appendices I and II of this report give seven possible sets of weights for each of the simple and the two-thirds majority cases. Tables I.7 and II.7 were the cases we recommended for the respective majorities. After some discussion the Board selected to use Table I.7 and II.2. Table II.2 was our second choice for the two-thirds majority, but the Board did not know that in advance. (Note that the final zero in the weight column C in Table II.2 can be deleted.) The judge then approved these selections and the weighted voting plan was introduced starting in January 1982.

Tompkins County also came up with several possible plans for redistricting into more equal sized districts, and will likely need to undertake redistricting in a few years. There also exists several mathematical approaches to this problem in the literature. It should be noted that some counties (e.g., Cortland) redistrict to within 5% and then do weighted voting as well.

WEIGHTED VOTING IN TOMPKINS COUNTY

1. Introduction. Reapportionment for county boards in New York State has been achieved by means of the following three methods.

(1) Redistricting the representative regions by altering their boundaries so that each such election district has nearly equal population.

(2) Weighted voting (also called "computerized" voting) in which each representative's vote is weighted in such a manner that his or her resulting power relates to the number of people represented.

(3) Multimember districts in which more than one representative may be elected at-large from the same district. A variant of this is the floterial case in which some members are elected from smaller districts whereas others may be elected at large from larger regions that cover or overlap some of the smaller districts.

This report is concerned with weighted voting in Tompkins County, New York in the early 1980s using the existing election districts and the 1980 U.S. census data. Some preliminary remarks about weighted voting, measuring power, and weighted voting in counties in New York State appear in sections 2,3 and 4. Some discussion about good solutions and computational aspects appears in sections 5 and 6. Several possible weighted voting systems for Tompkins County for the case of a simple majority of the board and for the case of a two-thirds majority are given in the tables appearing in appendices I and II, respectively. Explanations for these tables and recommendations appear in sections 6 and 7. Some additional documentation and computer programs appear in appendix III.

2. Weighted Voting. Weighted voting is frequently used when there is sufficient reason to create or maintain districts which have nontrivial variations in populations. This may be caused, e.g., by a desire to maintain some correspondence between election districts and existing municipalities or natural geographical boundaries. More than one-third of the counties in New York State have made use of such weighted voting systems. The object of such systems is to create equity by weighing the representatives' votes so as to properly adjust for the fact that they

represent different numbers of constituents.

In a weighted voting system there are n voters (representatives) denoted by

$$1, 2, 3, \ldots, n;$$

each voter has a corresponding weight (positive number)

$$w_1, w_2, w_3, \ldots, w_n;$$

and there is a quota q necessary to pass an issue. The number q may be a simple majority, a two-third majority, etc. A coalition (subset) of voters can pass a resolution if and only if their total weight is at least q; i.e., if the sum of the weights of the voters in this coaltion adds to q or greater. Such a coalition is called winning. All other coalitions are losing. Note that the role of the weights w_1, w_2, ..., w_n and quota q is to determine which coalitions of voters are winning.

One common and rather intuitive way to attempt to correct for districts of unequal population is to weigh the representatives' votes so that they are directly proportional to the number of people they represent. This approach frequently produces good results in terms of the relative powers of the representatives. However, there can arise cases in which the relative weight of one's vote is not a good measure of one's power. E.g., if four voters 1, 2, 3 and 4 had the weights $w_1 = 43$, $w_2 = 42$, $w_3 = 9$ and $w_4 = 6$ with a simple majority of $q = 51$ necessary to win, then the first three voters have equal power in affecting decisions whereas voter 4 has no power at all. So it is not always sufficient to weigh representatives' votes directly proportional to their respective populations.

Instead, one must introduce an intermediate step. One should assign weights to voters so that their resulting "power" to influence decisions is proportional to the number of people being represented. In New York State the courts accept a measure for power introduced by John F. Banzhaf III, a lawyer currently at Georgetown University, as an appropriate measure to use in such weighted voting situations. Therefore, the goal is to assign representatives weights so that their resulting Banzhaf power indices are directly proportional to their respective populations.

3. Power Indices. In general the notion of power is a difficult concept to quantify. However, the ability to pass resolutions in a given weighted voting system does exhibit some quantitative structure and properties, and several measures for such power have been proposed. The first serious measure of this type was the Shapley-Shubik index (or Shapley value (1953)). The one with legal precedence in the courts is the Banzhaf index (also called the Banzhaf-Coleman value). For many voting situations these two measures give quite similar results, but this is not always the case.

The Banzhaf power index is defined as follows. Consider all theoretically possible ways that the group of n voters can be partitioned into two coalitions, one in favor of an issue and the other against it. I.e., all possible combinations of the n voters into "yea" and "nay" groups. The number of such distinct combinations is 2^n, i.e., the same as the number of subsets of n elements. A given voter is decisive (or is called a swing) in a particular combination if by changing his or her vote (from yea to nay), while all other votes remain unchanged, the result of the group's vote will be reversed (from pass to fail). One also refers to such a voter in this particular combination as pivotal, critical or marginal. The Banzhaf index merely counts for each voter the number of combinations in which he or she is a swing. For any two voters it is only the relative magnitude of their number of swings which is important in comparing their power. So one can multiply such indices by a common constant. It is thus natural to express the Banzhaf index in a normalized form in which the sum of the indices for the n voters is a convenient constant such as 1 or n. (In the tables in this report the Banzhaf indices are normalized so that they sum to $n = 15$.)

The Banzhaf index has a probabilistic interpretation. If each combination of "yea" and "nay" votes is equally likely to occur, then a voter's index is proportional to the probability he will be decisive in a given vote. The Banzhaf index also has a more theoretical definition in terms of axioms. The more mathematically mature reader can consult the paper by Dubey and Shapley given in the references. Elementary explanations of the Banzhaf index along with several illustrations appear in the three papers by him, and in the one paper by Johnson, listed in the references.

4. New York State. The federal courts have ruled that the "one person, one vote" concept applies to local governments, school district trustees, and political party structures. Some brief highlights of early court dicisions in New York State regarding weighted voting and "one person, one vote" follows.

1966 In 1967, the Court of Appeals of New York ruled (Graham v. Board of Supervisiors of Erie County) that weighted voting may be approved "solely as a temporary expedient; but that a permanent plan must be based on the principle of 'one man, one vote'". At that time weighted voting could only be employed as a stopgap measure, to be used while a new apportionment system was being instituted. A year later, in a landmark case (Town of Greenburgh v. Board of Supervisors) the New York Court of Appellate Appeals upheld the constitutionality of a weighted voting system based on Division population. By the mid 1970s, at least twenty-two of the fifty-seven in 1968 counties of New York (outside of New York City) had adopted weighted voting schemes for their county board of legislators. However to conform with the ruling of the Court of Appeals ruling in Iannucci v. Board of Supervisors (1967), a "computerized analysis" must be presented to validify the Apportionment plan.

> "It is improper for a court in passing upon a constitutional question, to lightly disregard the considered judgement of a legislative body which is also charged with duty to uphold the Constitution but with respect to weighted voting a considered judgement is impossible without computer analysis and, accordingly if county board of supervisors chose to reapportion themselves by use of weighted voting there is no alternative but to require them to come forward with such analysis and demonstrate the validity of their apportionment plan."

A detailed discussion of the early history of weighted voting in county boards in New York State is given in the 1969 reference by Johnson.

Many of the counties in New York State have employed the consultant Lee Papayanopoulos, a professor in the School of Business at Rutgers University, to do the necessary computer calculations for their weighted voting plans. However, the mathematical or computer scientist so inclined can perform or check such calculations if they have an appropriate computer or calculator available.

Many examples of weighted voting in several other contexts appear in the references by Lucas and by Barrett and Newcombe.

5. Calculations and Approximations. The objective of a weighted voting system is to have the voters' (i.e., representatives') powers proportional to the populations they represent. The goal of this report is to determine what weights should be selected for the representatives so that their resulting Banzhaf indices will be proportional to the respective populations for the election districts in Tompkins County.

Two problems of a computational nature arise. First, not all n-tuples (of positive numbers that sum to 15) can arise as a (normalized) Banzhaf index. So no theoretically perfect answer may be attainable for a given set of population data. One must resort to good approximations to an ideal solution, i.e., weights for which the resulting Banzhaf indices are close to being proportional to respective populations. Second, there is no computationally feasible way to solve for the "best" set of weights, given the desired Banzhaf indices. So one must resort to doing the problem in the opposite order. One first "guesses" at a set of reasonable weights and then computes the corresponding indices, and examines the latter to see if they give a good approximation. Solving for indices, given the weights, is a feasible (though long) calculation. This latter process of selecting weights and computing indices normally has to be repeated many times, until one arrives at weights for which the indices are good approximations to the desired powers.

Weighted voting has frequently been referred to as "computerized voting" because the necessary calculations (for all but small values of n) requires a computer or programmable calculator. For 15 voters one must consider $2^{15} = 32,768$ combinations, and each voter will be decisive in a few thousand such combinations. So for each set of weights (and quota q) selected, one must perform many thousands of (simple) arithmetical operations to determine the Banzhaf indices. And this process is normally repeated many times in order to test many potential sets of weights.

The problem is considered as solved when one arrives at weights for which the resulting Banzhaf indices are acceptably close to being proportional to the respective populations. This closeness can be measured by comparing the fraction of swings assigned to a particular voter with the fraction of the county population residing in his or her district. In this report we normalize the Banzhaf indices so that they sum to $n = 15$, and we express district populations in terms of the average district population in Tompkins County. E.g., the Town of Lansing has a population of 8,317 which is 1.433 times the average district size of 5,806. We will say that the normalized population of Lansing is 1.433. The normalized Banzhaf index for Lansing should be close to this latter number. (This particular normalization has been selected, because recent reports on reapportionment in Tompkins County have expressed distict populations in terms of the average district population.)

One can calculate the difference between each voter's normalized index and the corresponding normalized population, and this number should be close to zero. Or else, one can examine the ratio of each voter's normalized index to his or her district's normalized population, and this number should be nearly one. A good solution should have all n differences (or ratios) within 1.5% of 0 (or 1, respectively). The courts would likely accept a plan which came within a few percentage points of an exact answer.

In column H of the tables in appendices I and II, the ratios of the (normalized) Banzhaf indices to the corresponding (normalized) populations are displayed. These are called the effectiveness factor. So that numbers between 0.9850 and 1.0150 correspond to less than 1.5% "error".

Algorithms for computing Banzhaf indices from given weights, and related computer programs, have been developed by many individuals. The calculations for this report made use of one similar to that described in the reference by Walther. The documentation for the computer aspects of the calculations in the report appears in appendix III.

6. Explanation of Tables. The tables in appendices I and II each have eight columns whose explanation is as follows. Recall that appendix I is for the case of a simple majority and appendix II is for a two-thirds majority.

A. The 15 districts for Tompkins County are listed in the order of decreasing population under the heading "Town". This order is for computational convenience, and allows for easier comparisons.

B. The corresesponding population figures for each district are given in the second column.

C. The weighted vote to be assigned to each representative appears in column C. At the bottom of this column there is the sum of the weights along with the quota necessary to win. The quota is just over half of the sum of the weights in appendix I, and equal to two-thirds of the total weight in appendix II. The sum of the weights is selected in I as an odd integer so as to avoid ties; i.e., situations in which there are two disjoint coalitions for which neither is winning.

D. There are $2^{15} = 32,768$ distinct combinations of "yea" and "nay" votes for a 15 member board in which every one votes. The number of such combinations in which a particular voter is decisive (and called a swing) is given in column D. Note that more than one voter may be a swing in some combinations, and that no voter may be decisive in some other combinations.

E. The resulting Banzhaf power indices are listed next. These indices are proportional to the numbers in column D. They are normalized here so that these sum to 15.

F. The district populations are listed in column F in terms of the average district size, and these are called normalized populations.

The closeness of the Banzhaf power indices to the populations can be determined by comparing columns E and F, and this is presented in two forms in the final two columns.

G. The numbers in column G come from those in columns E and F using the formula $G = 100(E - F)/F = 100((E/F) - 1)$. This is called the percentage discrepancy. One desires that such numbers be small; e.g., between -1.5 and $+1.5$. A positive number indicates that one has slightly more power than desired, and a negative number slightly less.

H. The final column gives the numbers $H = E/F$ and these ratios should be close to one. These numbers are called the effectiveness ratios.

In considering a good solution, one is concerned with several aspects of the numbers in columns G and H. These include the following. (1) The total (or global) percentage discrepancy which is the sum of the absolute values of the numbers in column G. This number is given below each table. (2) The (more local) largest individual discrepancies, either positive or negative, in column G. These also appear below each table. (3) The distributional aspects of where the positive or negative numbers appear in column G. E.g., if the top numbers were all positive and the botton numbers were all negative, then a bias in favor of larger districts would be indicated. One wishes to somehow minimize such global or local discrepancies or bias.

However, no one simple criterion for a best solution can be entirely satisfactory.

It should also be noted that the solutions recommended in the next section are among the "best" in some sense of many cases examined to date. It is computationally infeasible to calculate indices for all possible good solutions. Although theoretically possible, one cannot in practice for $(n = 15)$ determine the optimal solution; and the concept of "optimal" itself is also relative to the particular statistical measures which one selects to measure "closeness" or "goodness" of fit.

7. Recommendations. Seven reasonable solutions for both the simple majority and the two-thirds majority cases appear in the tables in appendices I and II, respectively. Tables I.1 and II.1 are the cases for "nonadjusted" weights. Each of these is a reasonable solution, except that they have a serious discrepancy for just one district (Lansing). All of the other tables in appendix I have very small discrepancies. The tables in appendix II show somewhat larger discrepancies, but these are within acceptable bounds. (There are some mathematical reasons which lead one to believe that it may be very difficult or impossible to obtain

significantly better bounds for the case of two-thirds majority.

The recommended weights and quotas for the two cases are listed below. A coalition of voters wins when the sum of its member's weights is greater or equal to the quota. Details regarding these recommended cases appear in tables I.7 and II.7 respectively.

District Number	Simple Majority	Two-thirds Majority
1	261	84
2	246	82
3	298	88
4	274	85
5	238	81
6	404	94
7	240	81
8	306	88
9	240	81
10	241	81
11	270	85
12	224	80
13	210	79
14	333	92
15	214	79
Total weight	3999	1260
Quota to win	2000	840

In the case of a simple majority: no coalition of 6 or fewer members can win, some coalitions of 7 can win, some coalitions need 8 to win, and all coalitions of 9 or more win. In the case of a two-thirds majority: no coalition of 9 or fewer members can win, some coalitions of 10 win, and all coalitions of 11 or more win.

It is recommended that the rule for a quorum of 8 remain unchanged, even though not all coalitions of 8 members are winning. It is also recommended that weighted voting not be adopted in committee considerations. Committees are usually not fully representative and not subject to the "one man-one vote" requirement.

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