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PROOF SCHEMES AND LEARNING STRATEGIES OF ABOVE-AVERAGE MATHEMATICS STUDENTS

ABSTRACT. What patterns can be observed among the mathematical arguments above-average students find convincing and the strategies these students use to learn new mathematical concepts? To investigate this question, we gave task-based interviews to eleven female students who had performed well in their college-level mathematics courses, but who differed in the number of proof-oriented courses each had taken. One interview was designed to elicit expressions of what students find convincing. These expressions were categorized according to the proof schemes defined by Harel and Sowder (1998). A second interview was designed to elicit expressions of what strategies students use to learn a mathematical concept from its definition, and these expressions were classified according to the learning strategies described by Dahlberg and Housman (1997). A qualitative analysis of the data uncovered the existence of a variety of phenomena, including the following: All of the students successfully generated examples when asked to do so, but they differed in whether they generated examples without prompting and whether they successfully generated examples when it was necessary to disprove conjectures. All but one of the students exhibited two or more proof schemes, with one student exhibiting four different proof schemes. The students who were most convinced by external factors were unsuccessful in generating examples, using examples, and reformulating concepts. The only student who found an examples-based argument convincing generated examples far more than the other students. The students who wrote and were convinced by deductive arguments were successful in reformulating concepts and using examples, and they were the same set of students who did not generate examples spontaneously but did successfully generate examples when asked to do so or when it was necessary to disprove a conjecture.

KEY WORDS: above-average, learning strategies, proof, proof schemes, college mathematics

1. Proof Schemes and Learning Strategies

1.1. *Introduction*

Student learning and understanding of mathematical proofs has been a major focus of recent mathematics education research (e.g., Balacheff, 1991; Chazan, 1993; Hanna and Jahnke, 1993; Goetting, 1995; Simon and Blume, 1996; Harel and Sowder, 1998; Selden and Selden, 2003). Another major research focus has been on student learning of mathematical concepts (e.g., Tall and Vinner, 1981; Moore, 1994; Dahlberg and Housman

1997). In the current study, our goals were (1) to explore the strategies some above-average students use to learn new mathematical concepts and the arguments these students use to convince themselves and others of the truth of mathematical conjectures, and (2) to observe patterns and possible relationships among the strategies they used and the arguments they found convincing. Expressing concepts in different ways (i.e., reformulation), coming up with examples and non-examples of a concept (i.e., example generation), and using examples to develop conjectures and check their validity (i.e., example usage) are learning strategies whose use by a student may be related to that student's choice of deductive, empirical, or other arguments to prove or disprove conjectures.

1.2. *Proof schemes*

Bell (1978, p. 48) points out that mathematical proof is concerned "not simply with the formal presentation of arguments, but with the student's own activity of arriving at conviction, of making verification, and of communicating convictions about results to others." Harel and Sowder (1998) define a *proof scheme* to be the arguments that a person uses to convince herself and others of the truth or falseness of a mathematical statement. They characterize seven major types of proof schemes, grouped into the three categories of external conviction, empirical, and analytical proof schemes (Harel and Sowder, 1998; Sowder and Harel, 1998; Harel, 2002). We will now describe the working definitions that we used in the current study.

There are three types of *external conviction* proof schemes and, in each, students convince themselves or others using something external to themselves. In a *ritual* proof scheme, the convincing is due to the form of the proof, not its content. An *authoritarian* proof scheme is one in which the convincing is due to the fact that the teacher or the book or some other authority said it was so. In a *symbolic* proof scheme, the convincing is by symbolic manipulation, behind which there may or may not be meaning.

Empirical proof schemes can be either inductive or perceptual. A student with an *inductive* proof scheme considers one or more examples to be convincing evidence of the truth of the general case. In a *perceptual* proof scheme, the student makes inferences that are based on rudimentary mental images and are not fully supported by deduction, and she considers these inferences to be convincing to herself or others. Harel and Sowder (1998) noted, "The important characteristic of rudimentary mental images is that they *ignore transformations on objects or are incapable of anticipating results of transformations completely or accurately*" (p. 255).

Analytical proof schemes can be either transformational or axiomatic. In a *transformational* proof scheme, the student convinces others or is convinced by a deductive process in which she considers generality aspects, applies goal-oriented and anticipatory mental operations, and transforms images. An *axiomatic* proof scheme goes beyond a transformational one, in that the student also recognizes that mathematical systems rest on (possibly arbitrary) statements that are accepted without proof.

Note that individual students may display aspects of different proof schemes in different contexts. The same can be said of professional mathematicians who operate from analytical proof schemes in their areas of research expertise but may be convinced, for example, of the validity of Wiles' proof of Fermat's Last Theorem by the authority attributed to Wiles and other mathematicians (Horgan, 1993).

1.3. *Learning strategies*

Dahlberg and Housman (1997) used task-based interviews to investigate the strategies students use to learn a new mathematical concept. Eleven students were given a formal definition and then were asked to carry out a number of tasks that both measured and helped to develop their understanding of the new concept. In their study, the students who used *example generation* (producing one or more examples related to the concept) and *concept reformulation* (expressing the concept using pictures, symbols, or words different from the definition) were the ones best able to develop a correct and complete *concept image*, as characterized by Tall and Vinner (1981) and Moore (1994). The students who used example generation were the ones who were best able to identify the correctness of conjectures and provide explanations. The students who primarily reformulated concepts without generating examples were able to determine whether a given object was an example of the mathematical concept, but these students were more easily convinced of the validity of a false conjecture. Although example generation and concept reformulation were the most beneficial learning strategies for the students in this sample, *example usage* – the use of provided examples – was also a significant factor in eliciting some learning events. In the current study, Dahlberg and Housman's task-based interview instrument was used to identify the reformulation, example generation, and example usage learning strategies employed by our student participants.

1.4. *Proof schemes and learning strategies*

Two research studies have related students' proof-writing abilities to their abilities to (a) generate and use examples (Moore, 1994) and (b) reformulate (Selden and Selden, 1995). However, these studies examined example usage, example generation, or reformulation in the context of proof writing and verification, not in the context of learning a new mathematical concept. Even more importantly, these studies focussed on the students' abilities to write or recognize correct mathematical proofs, not on the students' proof schemes – the arguments they use to convince themselves or others.

In the current study, we examined the following questions: What proof schemes and learning strategies are exhibited by some above-average students? What patterns can be observed among the proof schemes expressed by these students, the learning strategies they used, and the number of proof-oriented courses they had taken?

To investigate these questions, we gave task-based interviews to students. One interview was designed to elicit expressions of what students find convincing. These expressions were categorized according to the proof schemes defined by Harel and Sowder (1998). A second interview was designed to elicit expressions of what strategies students use to learn a mathematical concept from its definition, and these expressions were classified according to the learning strategies described by Dahlberg and Housman (1997).

2. Methods

2.1. *Participants and tasks*

The participants in this study were eleven undergraduate mathematics majors at a women's college who had earned only A's and B's in their college mathematics courses. We included in our sample students with different amounts of experience in reading and writing proofs: Four students (with pseudonyms Carol, Cathy, Chris, and Claire) had taken no college-level proof-oriented course, three students (Becky, Beth, and Bonnie) had completed one such course, and four students (Alice, Amy, Anne, and April) had completed two or more of these courses. Each student participated in two hour-long videotaped task-based interviews, which are described in the following two sections.

TABLE I

Conjectures

1.	The sum of the three interior angles of any triangle is 180 degrees.
2.	If no angle of a quadrilateral is obtuse, then the quadrilateral is a rectangle.
3.	If $(a+b)^2$ is even, then a and b are even.
4.	The product of two negative real numbers is always a positive real number.
5.	A polynomial of degree three must have at least one real root.
6.	If A is a subset of C and B is a subset of C, then the union of A and B is a subset of C.
7.	If an operation * is commutative, then * is associative.

2.2. *Proof schemes interview*

For 40 minutes, the students examined seven conjectures (see Table I), stated whether each was true or false, and provided written proofs. For the remaining 20 minutes, each student was asked the following, for each conjecture: How certain are you that the conjecture is true or false? How convincing is your proof to you? How convincing would your proof be to a peer? How convincing would your proof be to a mathematician? These questions about student conviction were necessary to determine the students' proof schemes because a proof scheme, by definition, consists of the arguments that a person uses to *convince* herself and others of the truth or falseness of a mathematical statement.

By presenting students with conjectures rather than theorems, we encouraged them to convince both themselves and others of the validity or invalidity of each conjecture. In a standard interpretation of these conjectures, only Conjectures 3 and 7 are false. While true in a Euclidean setting, Conjectures 1 and 2 are false in a non-Euclidean setting, and Conjecture 5 is false if polynomial coefficients need not be real numbers. We used this contextual ambiguity in the follow-up interviews to try to elicit expressions of the axiomatic proof scheme when students were confronted with a non-Euclidean counterexample to Conjecture 1, for example.

All of our participants had completed high school geometry and algebra courses and at least one semester of college-level calculus. So, they had heard of all of the concepts used in the conjectures (although none had previously seen non-Euclidean geometry), but they differed in their experience with certain concepts (e.g., nonassociative operations). It should be noted that we did not expect students to provide proofs for all conjectures, and we requested that they focus on those conjectures for which they thought they could provide the most convincing proofs.

At least three of our conjectures have been used in previous studies. Hoyles (1997) asked secondary school students to judge how convinced they were by a variety of arguments (correct, incomplete, and incorrect) for Conjecture 1. Galbraith (1981) asked secondary school students about the validity of (a) a short deductive argument for Conjecture 2 based on a drawing of a convex quadrilateral and (b) a self-intersecting quadrilateral as a counterexample to Conjecture 2. Zaslavsky and Peled (1996) asked in-service and pre-service teachers each to provide at least one example to convince a student that Conjecture 7 is false.

2.3. *Learning strategies interview*

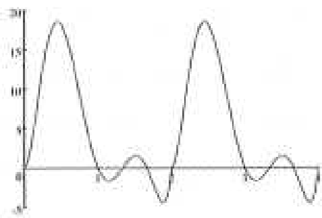
The instrument developed by Dahlberg and Housman (1997) for their study on learning strategies was used in this interview. In each interview, the student was presented with a new mathematical concept and was asked questions, both written and oral, involving the new concept. The interviewer tried to neither confirm nor dispute assertions made by the student; however, the interviewer would ask the student to explain her answers, and would try to provide opportunities for her to display each learning strategy. The interview was divided into five segments, each of which began with a page of information or questions being presented to the student.

The definition page gave the following definition: "A function is called *fine* if it has a root (zero) at each integer." Note that the graph of a fine function would include points at (0,0), ($\pm$1,0), ($\pm$2,0), and so forth, and otherwise need only satisfy the vertical line test (the graph contains no more than one point along any vertical line). Of course, a fine function may have any domain, including the complex numbers, as long as the domain includes every integer. We observed each student as she worked on the definition page and asked her, when she was ready to go on, what she had done to learn the new concept. Most students came up with one or more examples, provided a graphical interpretation, or provided a rewording of the definition.

Our choice of the fine function concept had four advantages. First, no student had seen the concept prior to the interview. Second, the base concepts on which the definition rests (i.e., *function*, *root*, *zero*, *at each*, and *integer*) were familiar to all students. Third, the base concepts *function* and *root* evoke rich and varied concept images among students (Breidenbach et al., 1992; Tall, 1992). Fourth, there was only a single quantifier, *at each*, which avoided the difficulties students often have with multiple quantifiers.

TABLE II

Examples page

1. $f(x) = \sin(\pi x)$
2. $f(x) = x^2 - x$
3. $f(x) = 0$
4. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$
5. $f(x) = \tan\left(\frac{\pi}{2}x\right)$
6. 

The questions page asked the student to provide an example of a fine function, an example of a function that was not fine, and an explanation in the student's own words and/or pictures of what a fine function is. Examples of fine functions cited by students included sinusoidal-looking graphs and symbolic examples such as $f(x) = \sin(\pi x)$ and $y = 0$. Reformulations included a variety of rewordings, graphical representations, and symbolic expressions such as $(n, 0)\forall n \in \mathbb{Z}$.

The examples page asked the student to determine whether each of the six functions given in Table II was fine or not. The sinusoidal, zero, and everywhere-discontinuous functions (Examples 1, 3, and 4) are fine. The functions in Examples 2 and 5 are not fine. The graphically represented function in Example 6 is not fine as drawn because it is not defined for integers outside of the interval [0,4]; however, it is fine if one assumes that the function is periodically extended over all real numbers. Either domain assumption, with the corresponding explanation, was acceptable to the interviewer, who would then follow up by asking the student to consider whether the function was fine with the other possible domain.

The conjectures page asked students to state whether, and explain why, each of the four conjectures given in Table III was true or false. All four conjectures are false: the zero function is a polynomial that is a fine function, $f(x) = \tan\left(\frac{\pi}{2}x\right)$ is a trigonometric function that is not fine, $f(x) = x\sin(\pi x)$ is an aperiodic fine function, and $f(x) = \sin(\pi x)\tan\left(\frac{\pi}{2}x\right)$ is the non-fine product of a fine function with another function. Notice that, except for Conjecture 3, counterexamples to the conjectures had already been presented to students on the prior (examples) page. If an opportunity presented itself, students were asked orally about the following (true)

TABLE III

Conjectures page

1.	No polynomial is a fine function.
2.	All trigonometric functions are fine.
3.	All fine functions are periodic.
4.	The product of a fine function and any other function is a fine function.

conjectures: (1) no nonzero polynomial is a fine function, and (2) the product of a fine function and any other function whose domain includes the integers is a fine function.

3. Data analysis

3.1. *Introduction*

The interview transcripts and written work from the proof schemes interviews and from the learning strategies interviews were analyzed qualitatively. We then compared the results from these analyses, observed patterns, and noted relationships between proof schemes and learning strategies. A student-by-student summary of our data analysis is presented in Table IV. An explanation of our classification system, illustrated by detailed descriptions of three student cases, is provided in the following sections.

3.2. *Proof schemes*

We used our seven working definitions (see Section 1.2) to identify expressions of one or more of the proof schemes in the written work and spoken remarks for each student-conjecture pair. We classified a proof scheme as *primary* for the student if the characteristics of the proof scheme were evident in the written proof and the expressed opinions for at least two conjectures or in the majority of the student's work. A proof scheme was considered *significant* for the student if the characteristics of the proof scheme were evident for one conjecture.

3.2.1. *Anne's proof schemes*

Anne exhibited primarily a transformational proof scheme with significant aspects of an axiomatic proof scheme. She was most convinced and thought others would be most convinced by (a) her correct deductive proof

TABLE IV

Data analysis

Student Pseudonym	External Conviction Proof Schemes: Authoritarian	External Conviction Proof Schemes: Ritual	External Conviction Proof Schemes: Symbolic	Empirical Proof Schemes: Inductive	Empirical Proof Schemes: Perceptual	Analytical Proof Schemes: Transformational	Analytical Proof Schemes: Axiomatic	Learning Strategies: Example Generation	Learning Strategies: Example Usage	Learning Strategies: Reformulation
Chris						primary	significant	P - N	moderate	moderate
Anne						primary	significant	P - N	successful	successful
Beth	significant		significant		primary	significant		P - N	successful	successful
Amy		significant			primary	primary		P - N	moderate	moderate
Claire					significant	primary		P - N	moderate	moderate
Alice					significant	primary		P - N	moderate	moderate
Carol				significant	primary			P U N	moderate	moderate
Becky			significant		primary			P U -	successful	successful
April					primary			P U -	moderate	moderate
Cathy		significant	primary		primary			P - -		
Bonnie		primary			significant			P - -		

Proof Scheme		Example Generation		Example Usage & Reformulation	
(dark)	primary	P	prompted (questions page)	(dark)	successful
(grey)	significant	U	unprompted (definition page)	(grey)	moderate
(blank)	insignificant	N	needed (conjectures page)	(blank)	unsuccessful

that the square of the sum of two odd numbers is even, a counter-argument for Conjecture 3, and (b) her element-chasing argument for Conjecture 6, which was correct except for a confusion between intersection and union of sets. She believed her correct deductive proof for Conjecture 1 was less convincing only because she invoked the Exterior Angle Theorem, which she believed was typically proved using Conjecture 1 rather than vice versa. This indicates Anne's concern for which statements are considered axiomatic. When the interviewer described a triangle formed on equator, Anne said that it was a counterexample to Conjecture 1 and pointed out the difference between 'plane' and '3-space' geometries, indicating that the concept of *line* could be different in different contexts (axiomatic systems). She recognized that her deductive proof of Conjecture 2 would not be as convincing to others because of an incomplete justification of a crucial step. For Conjecture 7, Anne considered addition, subtraction, multiplication, division, and some other operations in a failed search for a counterexample, started but was unable to complete a proof that the conjecture was true, and finally decided that she was unsure whether the conjecture was true or false. In short, Anne was convinced by deductive arguments, was unconvinced when a deductive argument was lacking, and, when grappling with Conjecture 1, exhibited evidence of an axiomatic proof scheme.

3.2.2. *Carol's proof schemes*

Carol provided arguments that Conjectures 1 and 2 were true. She was completely convinced by her argument for Conjecture 2: She began by defining and drawing pictures of various types of quadrilaterals, but she failed to consider all types of quadrilaterals. She next argued that

> by observing examples of each we see that a rhombus and a trapezoid may have obtuse angles. At times a rhombus, trapezoids, and squares can meet the qualifications of a rectangle when there are no obtuse angles ∴ when a quadrilateral has no obtuse angles it is considered a rectangle.

For Conjecture 1, she used casework based on what types of angles the triangle possessed, supporting most cases with incomplete pictorial arguments and one case with examples, but she was less convinced by her argument, noting that not all cases had been addressed.

Carol did not appeal to the authority of others nor did she use symbols in her arguments, and she spoke about the content of her proof rather than its style, so she does not have an external conviction proof scheme. Instead, Carol exhibited primarily a perceptual proof scheme with significant aspects of an inductive proof scheme: She convinced herself without a fully deductive argument and failed to see generalities inherent in the two conjectures (e.g., not considering all types of quadrilaterals). While she did use examples for Conjecture 2, the examples were used to prove an existence result rather than as evidence to support a generalization. For Conjecture 1, however, she used two examples as proof that the conjecture is satisfied by right triangles.

.2.3. *Cathy's proof schemes*

:athy primarily exhibited symbolic and perceptual proof schemes with ome aspects of a ritual proof scheme. Cathy was at least somewhat coninced by (a) her largely symbolic proof of Conjecture 4, in which the crux f the argument was that the "negatives cancel out," (b) her symbolic maipulation for Conjecture 5 that only returned her to the original problem, nd (c) her symbolic proof of Conjecture 6:

$$A \in C$$
$$B \in C$$
$$\therefore (A \subset B) \in C$$

n fact, if she was using the symbols $\in$ and $\subset$ to mean 'subset' and 'union,' espectively, then her work is nothing more than a symbolic restatement of ıe conjecture.

Cathy was very convinced by her seriously flawed arguments for Conectures 1 and 2. Below her drawings of five different triangles, a rectangle ith a diagonal, and a parallelogram with a diagonal, Cathy provided the ollowing proof for Conjecture 1:

ıny quadrilateral has the sum of interior angles = 360°. A triangle is formed by plitting a quad. with a diagonal. → The triangle will have half the degrees of a uadrilateral. ∴ The sum of the interior angles of any triangle is 180°.

'he proof starts incorrectly with an arbitrary quadrilateral rather than an rbitrary triangle and rests its case on the assertion, which is unsubstantited but reasonable based on her drawings, that the 'degrees' will divide venly between the two triangles formed by the quadrilateral's diagonal. 'or Conjecture 2, her proof starts incorrectly by assuming the conjecıre's hypothesis is false, later argues that the conjecture's converse is rue, and finally indicates that the truth of its converse proves the conjecıre. Cathy's convoluted arguments, which reveal difficulties with implication, the converse, and contraposition, could be interpreted as evidence or a ritual proof scheme. However, the manner in which her drawings elate to her written words and other evidence (e.g., Cathy indicated repeatedly that what mathematicians find convincing is related to form and ıat what convinces her is different) points more strongly to a perceptual roof scheme.

.3. *Learning strategies*

n the Learning Strategies Interview, we identified expressions of the exmple generation, example usage, and concept reformulation learning strategies, as done in the research study by Dahlberg and Housman (1997). ixample generation could occur with no explicit prompt on the definition page (U: *unprompted*), with explicit prompts on the questions page (P: *prompted*), and/or when needed on the conjectures page (N: *when needed*).

For example usage, a student was rated as *successful* if she used, on her own initiative, correct counterexamples for two conjectures and made some correct use of examples on another conjecture. A student was rated as *moderately successful* if she either (a) used a correct counterexample for one conjecture and made some correct use of examples on at least one other conjecture, or (b) made some correct use of examples on all four conjectures.

We recorded five different types of concept reformulation: verbal, symbolic, graphical, numerical, and factoring. A student was rated as *successful* for reformulation if she provided at least two correct reformulation types by the end of the questions page and at least three correct types by the end of the interview. A student was rated as *moderately successful* if she provided at least two correct types by the end of the interview.

3.3.1. *Anne's learning strategies*

Anne wrote nothing on the definition page and quickly moved to the questions page. She provided $y = x$ and its graph as an example of a non-fine function and a sinusoidal graph with integer roots as an example of a fine function. In addition to the graphical reformulation illustrated in these two examples, Anne symbolically reformulated fine function as "any function that is going to include the points $(n, 0)$ $\forall n \in \mathbb{Z}$." Anne correctly identified whether each example on the examples page was fine, including both interpretations of Example 6 (restricted to [0,4] or periodically extended). While working on Example 3, she stated that "the definition doesn't say that the function's only roots have to be all the integers," a verbal reformulation.

Anne stated that a "polynomial of infinite degree" would be a counterexample to Conjecture 1. When asked to consider only finite degree polynomials, she provided the zero function as a counterexample and then used a root-counting argument to explain why this would be the only such counterexample. Anne provided $\tan\left(\frac{\pi}{2}x\right)$ and $\sin(x)$ as counterexamples for Conjecture 2 and "an EKG-like graph" for Conjecture 3. She argued that Conjecture 4 was true with the implicit assumption that the other function was defined at all integers but conceded that the conjecture was false when presented with a counterexample.

In summary, Anne generated examples when prompted on the questions page and when necessary to disprove conjectures but not unprompted on the definition page. She used correct counterexamples to disprove three conjectures. She expressed correct graphical and symbolic reformulations on the questions page and a correct verbal reformulation on the examples page.

3.3.2. *Carol's learning strategies*

On the definition page, Carol drew several sinusoidal graphs with integer roots and a periodic discontinuous graph with integer roots. She said that she "was just trying to think of different possibilities that, at each integer, the function would cross the axis." Without prompting, Carol had graphically reformulated the concept and pursued an example generation strategy. On the questions page, Carol provided $y = x^2$ with its graph as an example of a non-fine function and $y = (x - 1)(x - 2)(x - 3)(x - 4)\ldots$ as an example of a fine function. She later stated that she had wanted to provide a sinusoidal function but could not find a formula for it. For the reformulation question, Carol wrote, "graphically the function will cross the x-axis at each integer ... [or] it may instead be tangent to the x-axis creating a root" and drew three graphs: (a) a sinusoidal graph vertically shifted so that its minimums lie on the integer coordinates of the x-axis, (b) a standard sinusoidal graph with integer roots, and (c) an aperiodic graph that transverses the x-axis at some integer coordinates and is tangent to the x-axis at the other integer coordinates. Even when prompted to reformulate, Carol spent most of her time coming up with new examples.

Aside from one error based on a graphical mistake, Carol correctly determined, usually with graphical explanations, whether each example on the examples page was fine, including both interpretations of Example 6. For Conjecture 1, Carol wrote, "when multiplied out $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)\ldots$ will be a high degree polynomial but it will still be fine," although she had second thoughts about whether this really was a function when the interviewer asked her to compute $f(1/2)$. She drew the graph of an aperiodic fine function as a counterexample to Conjecture 3 and made her second reformulation (verbal), noting that, "since there is a root at each integer, that doesn't mean those are the only roots in the function."

In summary, Carol generated examples unprompted on the definition page, when prompted on the questions page, and when it was necessary to disprove a conjecture. She used a correct counterexample to disprove one conjecture. She expressed a correct graphical reformulation on the questions page and a correct verbal reformulation on the conjectures page.

3.3.3. *Cathy's learning strategies*

On the definition page, Cathy wondered "is that saying, like, its root is x minus 1, x minus 2, x minus 3?" On the questions page, Cathy provided

$(x-1)(x-2)(x-3)$ as an example of a fine function "because its roots are consecutive integers" and $(x+5)(x-4)$ as an example of a non-fine function "since the roots are not at each integer." For the reformulation question, she summarized her incorrect understanding by writing, "A fine function is a function in which the roots are at consecutive integers." She applied this incorrect conception on the examples page until confronted with Example 6, when she decided that fine functions cannot have roots at nonintegers. Cathy applied both of her conceptions of the fine function (roots at consecutive integers, and no roots at nonintegers) to each example. When the interviewer pointed out that 2 was not a root of Example 2, Cathy decided that a fine function needs to have roots at every integer, not just at consecutive ones, but maintained her belief that fine functions cannot have roots at nonintegers. Cathy applied her new understanding consistently on all six examples.

Cathy said that Conjecture 1 was true because "a polynomial would have a finite set of roots, and it wouldn't go through every integer." When asked whether '3' is a polynomial, she incorrectly said no. Conjecture 2 was false "because all trigonometric functions don't have roots at zero for every integer," but Cathy did not offer a counterexample. Conjecture 3 was true "since it [a fine function] goes through every integer, it has a root at every integer; so that would be periodic." Conjecture 4 was false for if "the other function has roots that . . . aren't integers, then those would get added on also, so it would make it not fine." Cathy's answers and explanations are consistent with her understanding that (1) a fine function has a root at each integer and no roots at nonintegers, and (2) constant functions are not polynomials. However, she made no explicit use of examples or counterexamples in her explanations.

In summary, Cathy generated examples only when prompted on the questions page. She made no explicit use of examples or counterexamples when dealing with the conjectures. Although she made a number of incorrect or incomplete verbal reformulations, Cathy never expressed a completely correct reformulation until the very end of the interview when she verbally realized that there may be roots at nonintegers.

4. Discussion

4.1. *A small and select sample*

Our sample was small and select: eleven female post-secondary students who had received high marks in all of their college-level mathematics courses. As such, our findings cannot be generalized to other populations

On the contrary, our sample performed in ways that were clearly different from other student populations in at least two respects. First, every student in our sample successfully generated examples and nonexamples of fine functions, whereas Dahlberg and Housman (1997) observed fourth-year college students who were unable to successfully employ example generation. Second, only one student in our sample found an examples-based argument convincing (an expression of the inductive proof scheme), whereas Chazan (1993) found that at least one-third of the high school students that completed task-based interviews thought that measuring examples proved the general case, Martin and Harel (1989) found that more than 50% of the preservice elementary teachers they interviewed accepted examples-based arguments as mathematical proof, and Lewis (1986) found that 25% of Advanced Calculus students were satisfied with an examples-based argument on one question.

4.2. *Prior proof experience*

One might have expected that students who had not taken any proof-oriented courses would exhibit external conviction or empirical proof schemes, while students who had taken two or more proof-oriented courses would only exhibit analytical proof schemes. Surprisingly, however, each of the proof schemes categories (analytical, empirical, and external conviction) was exhibited by at least one student from each of the three levels of prior proof experience (indicated by the first letter of the pseudonym used for each student), as seen in Table IV.

Would students who had completed more proof-oriented courses be more successful in employing learning strategies? In our sample, the group of students that most successfully employed the example usage and reformulation learning strategies (Anne, Becky, and Beth) did not include a student with no prior proof course, and the pair of students that were least successful in employing example usage and reformulation (Bonnie and Cathy) did not include a student with more than one prior proof course. With regard to their example generation, students from each of the three levels of prior proof experience were found in each of these categories: (a) generated examples without prompting, (b) did not generate examples without prompting, (c) generated examples when necessary, and (d) did not generate examples when necessary.

4.3. *Multiplicity of proof schemes*

As Harel and Sowder noted, "A given person may exhibit various proof schemes during one short time span, perhaps reflecting her or his familiarity for, and relative expertise in, the contexts, along with her or his sense of

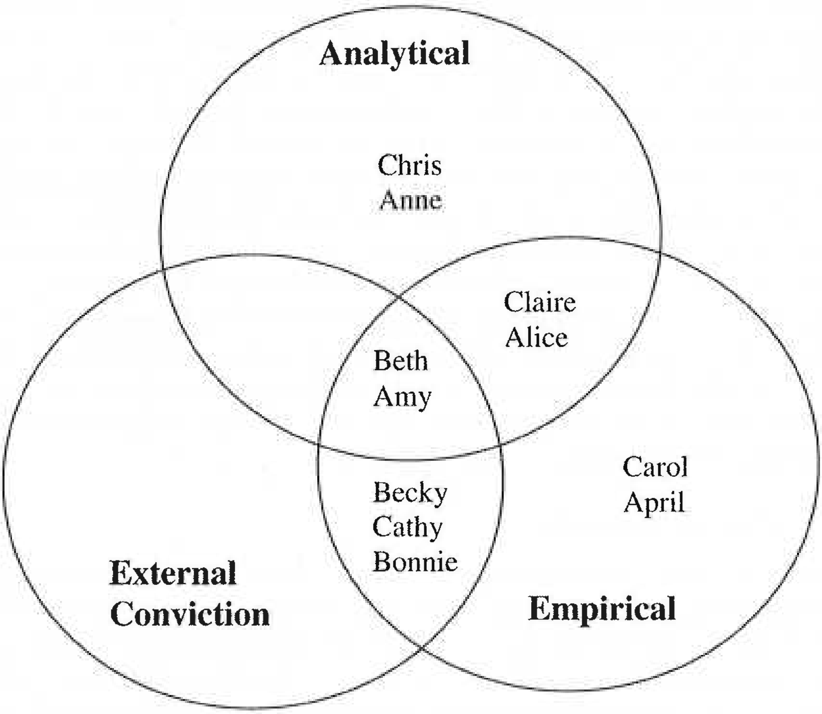


Figure 1. Student proof scheme categories exhibited.

what sort of justification is appropriate in the setting of the work" (1998, p. 277). Our data are consistent with this assertion. All but one of our eleven students were found to exhibit two or more proof schemes. One student exhibited four different proof schemes. Seven students were found to exhibit proof schemes in more than one of the major categories (external conviction, empirical, and analytical). As shown in Figure 1, students exhibited every combination of major categories of proof schemes except for (a) external conviction alone, and (b) external conviction and analytical. Because the students in this study were above average, it was not surprising that none exhibited exclusively external conviction proof schemes. Future research might find students who exhibit the combination of external conviction and analytical proof schemes or might suggest, for example, that there is a developmental path from external conviction through empirical to analytical proof schemes.

4.4. *Learning strategies*

For the students in our sample, success in employing the learning strategies was uniform in some respects (e.g., every student was able to generate examples when prompted to do so) and varied in other respects (e.g., there was at least one student in each of the four remaining example generation categories, which correspond to the four row groupings according to which Table IV is organized).

Interestingly, in our sample, each student's employment level (successful, moderate, or unsuccessful) of the example usage strategy was exactly the same as her employment level of the concept reformulation strategy. This differs from the Dahlberg and Housman (1997) sample, in which each student showed a clear preference for one learning strategy over the others.

4.5. *Proof schemes and learning strategies*

One might expect that students with weak employment of learning strategies would need to rely more on external sources to convince them of the validity of conjectures. In our sample, Bonnie and Cathy were the only students who had an external proof scheme as primary, and their employment of the three learning strategies was, in fact, weaker than that of the other students in our study. They generated examples only when explicitly prompted to do so; every other student generated examples in other circumstances. They were the only students who were unable to successfully generate or use examples on the conjectures page. Cathy gave only one correct reformulation and Bonnie never gave a correct reformulation; everyone else gave at least two different correct reformulations by the end of the interview.

Would a student who relied upon an example generation learning strategy exhibit primarily an empirical proof scheme? In our study, Carol, Becky, and April were the only students who generated an example unprompted, and they were also the only ones with the following combination: an empirical proof scheme as primary, no external conviction proof scheme as primary, and no analytical proof scheme. Carol, the only student to exhibit aspects of an inductive proof scheme, was also by far the strongest example generator, producing a multitude of examples unprompted, as well as when explicitly prompted and when otherwise needed.

Alice, Amy, Anne, Beth, Chris, and Claire were the students who displayed at least significant aspects of a transformational proof scheme. These are the same students who used example generation in moderation: they generated examples when explicitly prompted or when needed to disprove conjectures, but they did not generate examples unprompted. Carol was the only other student who generated examples when needed on the conjectures page, but she generated examples unprompted as well. The students with no significant aspects of a transformational proof scheme did not successfully generate examples when needed to disprove conjectures.

Thus, in our small and select sample, (a) the students with external proof schemes as primary were unsuccessful in their use of the example generation, example usage, and reformulation learning strategies; (b) the only student exhibiting aspects of an inductive proof scheme used the example generation learning strategy far more than the other students; and (c) the students with transformational proof schemes were at least moderately successful in their use of the reformulation and example usage learning strategies, and they were the same set of students who did not generate examples spontaneously but did successfully generate examples when asked to do so or when it was necessary to disprove a conjecture.

4.6. *Conclusion*

Before our study, little research had been done to investigate possible relationships among students' proof schemes, learning strategies, and prior proof experience. In our study, we have identified some patterns that existed among our students. Our study has also helped to generate some questions for future research: For above-average students, is there a general link between the successful employment of the reformulation and example usage learning strategies? Would instructing students in the use of the example generation learning strategy assist them in moving from an external conviction proof scheme to an empirical proof scheme? Would instructing students in the use of all three learning strategies help them to develop an analytical proof scheme? Investigating these questions would continue the work we have begun and would be helpful to educators as well as researchers.

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