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INFINITE PLAYER NONCOOPERATIVE GAMES AND THE CONTINUITY OF THE NASH EQUILIBRIUM CORRESPONDENCE*

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The usual definition of a noncooperative game is extended in two different ways: first, by replacing the finite player set with a measure space, and second, by eliminating the player set and considering a distribution of the players' characteristics. Feasible strategy profiles and Nash equilibria obtained from the two approaches are compared. The feasible strategy profile correspondence is shown to be continuous. The Nash equilibrium correspondence is shown to be upper hemicontinuous and nearly lower hemicontinuous on the class of convex and equicontinuous games. These results show when it is reasonable to use an infinite player game as an approximation of a large, but finite, player game.

1. Introduction. At times it is desirable to understand the strategic behavior of a large number of interacting agents. For example, the economy or political mechanisms of an entire country, commuter traffic patterns in a large city, or shareholder control of a large corporation. In such situations it is natural to posit a continuum of agents so that the mathematics is more tractable. Aumann's study of markets [1] is a classic example using cooperative games; Kannai and Peleg [11], Nti [14], and Housman [8] are examples using noncooperative games. An assumption inherent in such an approach is that the continuous player model accurately represents the large, but finite, player reality. Kannai [10] studied the validity of this assumption in the context of market games by considering the continuity properties of the core correspondence on the space of market games. This paper is in the spirit of [10], but is concerned with the Nash equilibrium correspondence on the space of noncooperative games.

We first generalize the definition of a finite player noncooperative game in two different ways so that games with an infinite number of players can be considered. In §3, this is accomplished by replacing the finite set of indices with a measure space (this is the approach taken by Schmeidler [15]). In §4, this is accomplished by considering the distribution of player characteristics (this is the approach taken by Mas-Colell [13]). §4 also develops the relationship between these two generalizations in a manner similar to the work of Hart, Hildenbrand and Kohlberg [7] on market economies. The models formulated here generalize those of Schmeidler [15] and Mas-Colell [13] in one significant respect: individual players can have a positive influence on other players' payoffs. Mathematically, we allow the measure space to contain singletons and one characteristic that a player possesses is her measure as a singleton. These more general models are useful when a few of the millions of stockholders of a corporation hold a significant proportion of the stock, or in an economy of oligopolist producers and many consumers.

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The continuity properties of the feasible strategy profile and Nash equilibrium correspondences are studied in §5. The Nash equilibrium correspondence is shown to be upper hemicontinuous and to have a weak lower hemicontinuity property. A general Nash equilibrium existence theorem, which is a generalization of [15] and [13], is also stated.

Green [6] presents a different generalization of finite player noncooperative games and proves that the Nash equilibrium correspondence thereby obtained is upper hemicontinuous. The author believes the present formulation and results to be more transparent and general than those in [6]. Also the existence and lower hemicontinuous results of this paper have no counterpart in [6].

2. Notation and mathematical preliminaries. Throughout this section, X denotes a complete separable metric space with the bounded metric d_X, or simply d if the space on which the metric is defined is clear from context. Note that any metric can be bounded by considering $\min\{1, d\}$ without changing the topology of X.

Let $B_\epsilon(w, d_X) = \{x \in X\colon d_X(w, x) < \epsilon\}$ and $B_\epsilon(W, d_X) = \{x \in X\colon d_X(w, x) < \epsilon$ for some $w \in W\}$. The symbols $B_\epsilon(w, d)$, $B_\epsilon(W)$, etc. are used when it is clear from the context what the other parameters are. $\bar{B}_\epsilon(w, d)$ and so forth denote the closure of $B_\epsilon(w, d)$ and so forth. Let $\operatorname{diam}(W) = \sup\{d(r, s)\colon r, s \in W\}$.

$\mathscr{F}(X)$ denotes the space of nonempty, closed subsets of X topologized by the Hausdorff metric $d(K, L) = \inf\{\epsilon > 0\colon K \subseteq B_\epsilon(L, d_X)$ and $L \subseteq B_\epsilon(K, d_X)\}$. $\mathscr{F}(X)$ is complete.

$\mathscr{K}(X)$ denotes the space of nonempty, compact subsets of X topologized by the Hausdorff metric. $\mathscr{K}(X)$ is complete and separable; if X is compact, then $\mathscr{K}(X)$ is compact [3, p. 354].

$\mathscr{B}(X)$ denotes the Borel subsets of X. Suppose $(A, \mathscr{A}, \alpha)$ is a probability space (i.e., $\mathscr{A}$ is a σ-field, and α is a countably additive, nonnegative, real-valued function on $\mathscr{A}$ satisfying $\alpha(A) = 1$) with $\mathscr{A}$ containing all singletons of A. If $a \in A$, then $\alpha(a)$ denotes $\alpha(\{a\})$, and we shall consider α as a real-valued function on A, on occasion. A function $f\colon A \to X$ is *measurable* if $\forall W \in \mathscr{B}(X)\colon f^{-1}(W) \in \mathscr{A}$; it is sufficient to verify for $W \in \mathscr{F}(X)$. $X(A, \mathscr{A}, \alpha)$ denotes the space of all measurable functions $f\colon A \to X$. A function $F\colon A \to \mathscr{F}(X)$ is *lower measurable* if $\forall W \in \mathscr{F}(X)\colon F^{-}(W) = \{a \in A\colon F(a) \cap W \neq \varnothing\} \in \mathscr{A}$. We shall make use of the Measurable Selection Theorem due to Kuratowski and Ryll-Nardzewski [12]: if $F\colon A \to \mathscr{F}(X)$ is lower measurable, then there exists a measurable $f\colon A \to X$ satisfying $f(a) \in F(a)$ for all $a \in A$.

$\mathscr{C}(X)$ denotes the space of bounded, continuous real-valued functions on X topologized by the sup norm $|f| = \sup\{|f(x)|\colon x \in X\}$. $\mathscr{C}(X)$ is complete [5, p. 261]. If X is compact, then $\mathscr{C}(X)$ is separable [5, Exercise V.7.17].

$\mathscr{M}(X)$ denotes the space of probability measures on X endowed with the usual weak topology, i.e., a base of open neighborhoods consists of sets of the form $N(\mu, f, \epsilon) = \{\nu \in M(X)\colon |\int f d(\mu - \nu)| < \epsilon\}$ for $\mu \in \mathscr{M}(X)$, $f \in \mathscr{C}(X)$, and $\epsilon > 0$. This topology is metrizable by the Prohorov metric $\rho(\mu, \nu) = \inf\{\epsilon > 0\colon \mu(E) \leqslant \nu(E^\epsilon) + \epsilon$ and $\nu(E) \leqslant \mu(E^\epsilon) + \epsilon$ for all $E \in \mathscr{F}(X)\}$ where $E^\epsilon = B_\epsilon(E, d_X)$ [2, Theorem 5, p. 238]. A subset $M \subseteq \mathscr{M}(X)$ is called *tight* if $\forall \epsilon > 0\ \exists K \in \mathscr{K}(X)\ \forall \mu \in M\colon \mu(X \setminus K) < \epsilon$. M is tight if and only if M is relatively compact [2, p. 37]. $\mathscr{M}(X)$ is a convex subset of the locally convex linear topological space of signed, countably additive set functions on the Borel sets of X with vector addition and scalar multiplication defined in the natural way: $(\mu + \nu)(E) = \mu(E) + \nu(E)$ and $(c\mu)(E) = c\mu(E)$. If $\mu \in \mathscr{M}(X)$, then let $\operatorname{supp} \mu = \{x \in X\colon \mu(B_\epsilon(x, d_X)) > 0$ for every $\epsilon > 0\}$. Let $p(x)$ denote the point measure at x, i.e., $p(x)(E) = 1$ if $x \in E$, and $p(x)(E) = 0$ if $x \notin E$.

Suppose W is a metric space. The function F: $W \to \mathcal{F}(X)$ is said to be *upper hemicontinuous*, *lower hemicontinuous*, and *continuous* at w, respectively, if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_W(w, w') < \delta$ implies, respectively,

(1) $B_\epsilon(F(w), d_X) \supseteq F(w')$,

(2) $B_\epsilon(F(w'), d_X) \supseteq F(w)$, and

(3) $d(F(w), F(w')) < \epsilon$.

Note that some authors use the prefix "semi" instead of "hemi"; our terminology is consistent with [7].

F is continuous if and only if it is both upper and lower hemicontinuous. F is said to have a *closed graph* at w if $d_W(w_n, w) \to 0$, $x_n \in F(w_n)$, and $d_X(x_n, x) \to 0$ implies that $x \in F(w)$. If F has a closed graph at w and $F(w)$ is compact, then F is upper hemicontinuous at w.

If $\sigma \in \mathcal{M}(W \times X)$, then the marginal distribution of σ on W is denoted $\sigma|_W$, i.e., $\sigma|_W(E) = \sigma(E \times X)$. $d_{W\times X}((w_1, x_1), (w_2, x_2)) = \sup\{d_W(w_1, w_2), d_X(x_1, x_2)\}$. If $\sigma(E \times F)$ is defined and countably additive for all $E \in \mathcal{F}(W)$ and $F \in \mathcal{F}(X)$, then σ can be extended uniquely to a measure on $W \times X$. If f: $A \to W$ and g: $A \to X$, then $f \times g$: $A \to W \times X$ is defined by $(f \times g)(a) = (f(a), g(a))$.

The logical qualifier symbols $\forall$ and $\exists$ are used in proofs. Also, "s.t." means "subject to," and "$\forall_\alpha a \in A$" means "for α-almost every $a \in A$."

3. Games with named players. In this section we generalize the usual definition of a strategic (or normal form) game by replacing the finite player set with a measure space.

DEFINITIONS. Suppose $(A, \mathcal{A}, \alpha)$ is a probability space with $\mathcal{A}$ containing all singletons of A and S is a nonempty compact metric space. If β: $A \to \mathcal{K}(S)$ and π: $A \to \mathcal{C}(S \times \mathcal{M}(S))$ are measurable, then (β, π) is a *game with named players* on $(A, \mathcal{A}, \alpha, S)$. We will use $\pi(a, s, \eta)$ to denote $\pi(a)(s, \eta)$. A *feasible strategy profile* of the game (β, π) is an $\mathrm{s} \in S(A, \mathcal{A}, \alpha)$ satisfying $\forall_\alpha a \in A$: $\mathrm{s}(a) \in \beta(a)$. A *Nash equilibrium* of the game (β, π) is a feasible strategy profile s satisfying $\forall_\alpha a \in A$ $\forall r \in \beta(a)$: $\pi(a, \mathrm{s}(a), \alpha \circ \mathrm{s}^{-1}) \geqslant \pi(a, r, \alpha \circ \mathrm{s}^{-1} + \alpha(a)[p(r) - p(\mathrm{s}(a))])$. The set of all feasible strategy profiles and Nash equilibria of the game (β, π) are denoted by $\Lambda(\beta, \pi)$ and $\Phi(\beta, \pi)$, respectively.

REMARK 1. A "play" of the game can be interpreted as follows. Each player $a \in A$ chooses a strategy $\mathrm{s}(a)$ among her feasible strategies $\beta(a)$ and then receives a payoff of $\pi(a, s(a), \alpha \circ s^{-1})$. The payoff to a depends on the player's own strategy $\mathrm{s}(a)$ directly as well as through the aggregate distribution $\alpha \circ \mathrm{s}^{-1}$ so that players of zero measure can still have an effect on their own payoff. Of course, for the payoff to be defined, s must be measurable. The measurability of β and π and the Measurable Selection Theorem [12] assure us that such a feasible strategy profile s exists. Nonetheless, the requirement that players choose their strategies in a jointly measurable fashion can be thought of as a restriction upon the independence of the individual players. Dubey and Shapley [4] introduced strategic games in "coalitional strategic" form as one possible justification for this weak form of coordination. This paper does not seek further justification than the usefulness of the concept of an infinite player game. The definition of Nash equilibrium has been extended in a natural fashion. Usually some sort of convexity or nonatomic properties are invoked to insure the existence of Nash equilibria (see [15], [13], [9]). Note that for players of zero measure, i.e., $\alpha(a) = 0$, the inequality given can be written $\pi(a, \mathrm{s}(a), \alpha \circ \mathrm{s}^{-1}) \geqslant \pi(a, r, \alpha \circ \mathrm{s}^{-1})$.

REMARK 2. The usual finite player strategic game $[N; K_1, \ldots, K_n; f_1, \ldots, f_n]$, where $N = \{1, \ldots, n\}$, the K_i are compact metric spaces, and the $f_i \in \mathcal{C}(K_1 \times \cdots \times K_n)$, can be equivalently represented by a game with named players. Indeed, let $(A, \mathscr{A}, \alpha)$ be the uniform probability measure space on N; let $S = N \times (K_1 \cup \cdots \cup K_n)$; let $\beta(i) = \{i\} \times K_i$; and let $\pi(i) \in \mathscr{C}(S \times \mathscr{M}(S))$ be an extension (guaranteed by Tietze's Theorem [17, Theorem 15.8]) of $g(i)(s, \eta) = f_i(x_1, \dots, x_n)$ if $\eta = (1/n)[p(1, x_1) + \cdots + p(n, x_n)]$ and $s = (i, x_i)$.

REMARK 3. Λ and Φ are correspondences from the space of games to the space of all strategy profiles $S(A, \mathscr{A}, \alpha)$; hence, it becomes possible to consider the continuity properties of these correspondences once the two spaces are given suitable topologies. Both spaces are function spaces and so there is no unique "natural" topology to choose. Housman [9] considers a general class of topologies and the continuity properties obtained.

REMARK 4. It is easy to further extend the definition of game by allowing players' payoffs to be dependent on the entire feasible strategy profile $\mathbf{s}$ rather than its distribution $\alpha \circ \mathbf{s}^{-1}$. This is considered by Housman [9] and Schmeidler [15]. Housman considers various topologies on the space of feasible strategy profiles that are no weaker than pointwise convergence; Schmeidler considers the L_1 weak topology and $(A, \mathscr{A}, \alpha)$ is assumed to be nonatomic. The restriction to payoff dependent on $\alpha \circ \mathbf{s}^{-1}$ is weaker than what one might initially guess (see Remark 2 and [13, Remark 1]); however, the restriction has content. The reason for the restriction is so that comparisons can be made with a different definition of game to be given in the next section. We can allow S to be a complete metric space with bounded metric rather than compact; this is considered in Housman [9].

4. Games without named players. In a game with named players, a player is given a name $a \in A$ but is characterized by her strategy set $\beta(a)$, a payoff function $\pi(a)$, and a size (of influence over the other players) $\alpha(a)$. In this section we develop an approach to strategic games in which the players are characterized by a strategy set, payoff function, and size without assigning names to players.

DEFINITION. Suppose S is a compact metric space. Let $\mathscr{T}(S) = \mathscr{K}(S) \times \mathscr{C}(S \times \mathscr{M}(S)) \times [0, 1]$, and $\mathscr{T}_0(S) = \{(K, f, 0) \in \mathscr{T}(S)\}$. A *game without named players* on S is a $\mu \in \mathscr{M}(\mathscr{T}(S))$ satisfying $\forall (K, f, x) \in \operatorname{supp} \mu \setminus \mathscr{T}_0(S)$: $\mu(K, f, x)/x$ is a positive integer. The space of games without named players on S is denoted by $\mathscr{G}(S)$.

A game without player names gives an accounting of the number (or measure) of players of each possible "type." A "type" is a $(K, f, x) \in \mathscr{T}(S)$ where K is a strategy set, f is a payoff function, and x is a size. If a type has a nonzero size, then there must be a nonnegative integral number of players of that type in any particular game which leads to the one restriction on μ in the definition.

REMARK 5. All of the results in this paper can be extended to the case when S is a complete metric space with a bounded metric. $\mathscr{M}(X)$ is redefined to be the space of Borel probability measures with separable support (in the present case, $\mathscr{T}(S)$ is separable so that all probability measures on $\mathscr{T}(S)$ have separable support), and Varadarajan [16] should be consulted instead of Billingsley [2]. The statement of all results would be the same except for Theorem 7 (see Remark 7).

REMARK 6. Unlike Mas-Colell [13], this formulation of game allows for players to be of nonzero size and to have different strategy sets. The value of such a generalization was discussed in the introduction. It should be noted that most nonatomic games under the present definition have representations as games under Mas-Colell's definition. The two representations will have identical equilibria; however, infeasible strategy profiles under the present definition will become feasible strategy profiles and result in different payoffs under Mas-Colell's definition. Hence, the natural topological structures on spaces of games under the two definitions do not relate in an easily discernible way. The present definition seems better suited for questions of equilibria correspondence continuity in applications.

DEFINITIONS. A *strategy profile* on a compact metric space S is a $\sigma \in \mathscr{M}(\mathscr{T}(S) \times S)$ satisfying $\forall (K, f, x, s) \in \operatorname{supp} \sigma$ with $x > 0$: $\sigma(K, f, x, s)/x$ is a positive integer. The space of all strategy profiles on S is denoted by $\mathscr{S}(S)$. A *feasible strategy profile* for the game μ is a strategy profile σ satisfying $\sigma|_{\mathscr{T}(S)} = \mu$ and $\forall (K, f, x, s) \in \operatorname{supp} \sigma$: $s \in K$. The set of all feasible strategy profiles of the game μ is denoted by $\Lambda(\mu)$. Hence, Λ is a correspondence from $\mathscr{C}(S)$ to $\mathscr{S}(S)$.

A strategy profile gives an accounting of the number of players of each possible type choosing each possible strategy. So, a strategy profile carries with it the definition of a game in a marginal distribution. For a strategy profile to be feasible for a particular game, clearly the game defined by the strategy profile must be the game under consideration. Also players are restricted to using strategies in their own strategy set. Note that $\mathscr{S}(S)$ is closed.

DEFINITION. The feasible strategy profile σ is a *Nash equilibrium* for the game μ if $\forall (K, f, x, s) \in \operatorname{supp} \sigma\ \forall r \in K$: $f(s, \sigma|_S) \geqslant f(r, \sigma|_S + x[p(r) - p(s)])$. The set of all Nash equilibria of the game μ is denoted by $\Phi(\mu)$. Hence, Φ is a correspondence from $\mathscr{C}(S)$ to $\mathscr{S}(S)$.

We now consider the relationship between games with and without named players. Suppose (β, π) is a game with player names on $(A, \mathscr{A}, \alpha, S)$ and $\mathrm{s} \in \Lambda(\beta, \pi)$. It is clear that $\mu = \alpha \circ (\beta \times \pi \times \alpha)^{-1}$ is a game without player names on S having the same distribution of player types as (β, π), and $\sigma = \alpha \circ (\beta \times \pi \times \alpha \times \mathrm{s})^{-1} \in \Lambda(\mu)$. If $\mathrm{s} \in \Phi(\beta, \pi)$, then $\sigma \in \Phi(\mu)$. Going from games without named players to games with named players is not as clear-cut, because two different games with named players may have identical player type distributions. Example 1 shows that there may be a feasible strategy profile (or Nash equilibrium) in a game without player names that does not correspond to any feasible strategy profile (or Nash equilibrium) in a game with player names having the same distribution of player types. Nonetheless, the relationship is shown to be very close by Theorem 1. Further, Example 2 shows that there is a perfect representative of a game without named players by a game with named players. Similar results for cores of cooperative games arising from market economies were first shown by Hart, Hildenbrand and Kohlberg [7].

EXAMPLE 1. Let $A = [-1/2, 1/2]$, $\mathscr{A}$ the σ-field of Lebesgue measurable subsets of A, α the Lebesgue measure on $(A, \mathscr{A})$, and $S = \mathbb{R}$. Let $\beta_1(a) = [0, 3/2 + a]$, $\beta_2(a) = [0, 1 + 2|a|]$, and $\pi(a)$ be the zero function for all $a \in A$. Clearly, $\alpha \circ (\beta_1 \times \pi \times \alpha)^{-1} = \alpha \circ (\beta_2 \times \pi \times \alpha)^{-1} = \mu$. Consider s defined by $\mathrm{s}(a) = 1 + 2|a|$ if $0 < a$, and $\mathrm{s}(a) = 0$ if $a \leqslant 0$. Clearly, $\mathrm{s} \in \Lambda(\beta_2, \pi)$, and so $\sigma = \alpha \circ (\beta_2 \times \pi \times \alpha \times \mathrm{s})^{-1} \in \Lambda(\mu)$. However, there is no $\mathrm{r} \in \Lambda(\beta_1, \pi)$ for which $\alpha \circ (\beta_1 \times \pi \times \alpha \times \mathrm{r})^{-1} = \sigma$. Similar remarks hold for Nash equilibria since $\Phi(\beta_i, \pi) = \Lambda(\beta_i, \pi)$ for $i = 1, 2$.

THEOREM 1. *Suppose (β, π) is a game with player names on $(A, \mathscr{A}, \alpha, S)$ and $\mu = \alpha \circ (\beta \times \pi \times \alpha)^{-1}$. If $\overline{D}(E)$ is the closure of $\{\alpha \circ (\beta \times \pi \times \alpha \times \mathrm{s})^{-1}: \mathrm{s} \in E\}$, then $\Lambda(\mu) = \overline{D}(\Lambda(\beta, \pi))$ and $\Phi(\mu) = \overline{D}(\Phi(\beta, \pi))$.*

PROOF. The remarks before the theorem show that $\overline{D}(\Lambda(\beta, \pi)) \subseteq \Lambda(\mu)$ and $\overline{D}(\Phi(\beta, \pi)) \subseteq \Phi(\beta, \pi)$.

Suppose $\sigma \in \Lambda(\mu)$ and $\epsilon > 0$. Since S is compact, there exists a finite partition $\{S_n\}$ of S for which $\operatorname{diam}(S_n) < \epsilon$ for each n. Since σ is nonnegative and finite, the S_n can be chosen so that if $\sigma(\mathscr{T}(S) \times \{s\}) > 0$, then there is an n for which $S_n = \{s\}$. Let

$A_n = \{a \in A \setminus \bigcup_{m=1}^{n-1} A_m: (\beta(a), \pi(a), \alpha(a)) \in \operatorname{supp} \sigma(\cdot \times S_n)\}$. Since β and π are measurable, $A_n \in \mathscr{A}$. Let $\gamma: A \to \mathscr{F}(S)$ be defined by $\gamma(a) = \beta(a) \cap \overline{B}_\epsilon(S_n)$ if $a \in A_n$. Clearly, γ is nonempty-valued and lower measurable. So, the Measurable Selection Theorem [12] implies that there exists a measurable function $s: A \to S$ satisfying $s(a) \in \beta(a) \cap \overline{B}_\epsilon(S_n)$ if $a \in A_n$. Hence, $s \in \Lambda(\beta, \pi)$. Let $\tau = \alpha \circ (\beta \times \pi \times \alpha \times s)^{-1}$. We now show that $\rho(\sigma, \tau) < 2\epsilon$. Suppose $E \in \mathscr{F}(\mathscr{T}(S))$ and $R \in \mathscr{F}(S)$. If $S' = \{s \in S: s \in S_n \text{ and } S_n \cap R \neq \varnothing\}$, then $\sigma(E \times R) \leqslant \sigma(E \times S') \leqslant \tau(E \times B_\epsilon(S')) \leqslant \tau(E \times B_{2\epsilon}(R)) \leqslant \tau(B_{2\epsilon}(E \times R)) + \epsilon$. If $S' = \{s \in S: s \in S_n \text{ and } B_\epsilon(S_n) \cap R \neq \varnothing\}$, then $\tau(E \times R) \leqslant \sigma(E \times S') \leqslant \sigma(E \times B_{2\epsilon}(R)) \leqslant \sigma(B_{2\epsilon}(E \times R)) + \epsilon$. Therefore, $\sigma \in \overline{D}(\Lambda(\beta, \pi))$.

Suppose $\sigma \in \Phi(\mu)$ and $\epsilon > 0$. Let $\gamma: \operatorname{supp} \mu \to \mathscr{K}(S)$ be defined by $\gamma(K, f, x) = \{s \in S: (K, f, x, s) \in \operatorname{supp} \sigma\}$. Since $\operatorname{supp} \sigma$ is a closed set, γ has a closed graph and so is measurable. So, $\beta' = \gamma \circ (\beta \times \pi \times \alpha)$ is measurable, and $\Lambda(\beta', \pi) = \Phi(\beta', \pi) \subseteq \Phi(\beta, \pi)$. Let $\sigma' \in \mathscr{S}(S)$ be defined by $\sigma'(E) = \sigma(\{(K, f, x, s): (\gamma(K, f, x), f, x, s) \in E\})$ and note that $\sigma'|_{\mathscr{T}(S)} = \alpha \circ (\beta \times \pi \times \alpha)^{-1}$. By the first paragraph, there exists a $s \in \Lambda(\beta', \pi)$ satisfying $\rho(\alpha \circ (\beta' \times \pi \times \alpha \times s)^{-1}, \sigma') < \epsilon$. But then $\rho(\alpha \circ (\beta \times \pi \times \alpha \times s)^{-1}, \sigma) < \epsilon$ and $s \in \Phi(\beta, \pi)$. Therefore, $\sigma \in \overline{D}(\Phi(\beta, \pi))$.

EXAMPLE 2. Suppose μ is a game without named players on S. Let $A = \mathscr{T}(S) \times [0,1]$, $\mathscr{A}$ be the Borel subsets of A, β be defined by $\beta(K, f, x, z) = K$, and π be defined by $\pi(K, f, x, z) = f$. If $I \in \mathscr{F}([0,1])$ and $E \in \mathscr{F}(\mathscr{T}_0(S))$, then let $\alpha(E \times I) = \mu(E)\lambda(I)$ where λ is Lebesgue measure. If $(K, f, x) \in \operatorname{supp} \mu \setminus \mathscr{T}_0(S)$, then let $\alpha(\{(K, f, x)\} \times I) = \mu(K, f, x)[|\{j: j/m \in I\}|/m]$ where $m = \mu(K, f, x)/x$. If $t \notin (\operatorname{supp} \mu) \cup \mathscr{T}_0(S)$, then let $\alpha(\{t\} \times I) = 0$. Clearly, $(A, \mathscr{A}, \alpha)$ is a probability space and (β, π) is a game on $(A, \mathscr{A}, \alpha, S)$ for which $\mu = \alpha \circ (\beta \times \pi \times \alpha)^{-1}$. It is straightforward (using the separability of A) to show that $\sigma \in \Lambda(\mu)$ implies that there exists $s \in \Lambda(\beta, \pi)$ for which $\sigma = \alpha \circ (\beta \times \pi \times \alpha \times s)^{-1}$. Thus, $\Lambda(\mu) = \{\alpha \circ (\beta \times \pi \times \alpha \times s)^{-1}: s \in \Lambda(\beta, \pi)\}$ and $\Phi(\mu) = \{\alpha \circ (\beta \times \pi \times \alpha \times s)^{-1}: s \in \Phi(\beta, \pi)\}$.

5. Existence and continuity results.

In this section we consider the nonemptiness and continuity of the feasible strategy profile and Nash equilibria correspondences. Any reasonable model for a strategic form game should have a nonempty-valued and continuous feasible strategy profile correspondence. That this is the case for games without named players is shown by Theorems 2 and 3.

THEOREM 2. *$\Lambda(\mu)$ is nonempty and compact.*

PROOF. Define $H: \mathscr{T}(S) \to \mathscr{K}(S)$ by $H(K, f, x) = K$. Clearly, H is nonempty-valued and measurable. By the Measurable Selection Theorem [12], there exists a measurable function $h: \mathscr{T}(S) \to S$ satisfying $h(K, f, x) \in K$. Now define $\sigma \in \mathscr{S}(S)$ by $\sigma(V) = \mu(\{t \in \mathscr{T}(S): (t, h(t)) \in V\})$. Clearly, $\sigma \in \Lambda(\mu)$.

Fix $\epsilon > 0$ because $\{\mu\}$ is a compact set, it is also tight. So, there exists a compact $T \subseteq \mathscr{T}(S)$ satisfying $\mu(T) > \mu(\mathscr{T}(S)) - \epsilon$. Now $R = \{s \in S: s \in K, (K, f, x) \in T\}$ is compact, and $\sigma \in \Lambda(\mu)$ implies that $\sigma(T \times R) = \sigma|_S(T) = \mu(T) > \mu(\mathscr{T}(S)) - \epsilon = \sigma(\mathscr{T}(S) \times S) - \epsilon$. Hence, $\Lambda(\mu)$ is tight and therefore relatively compact. Finally, $\Lambda(\mu)$ is clearly closed.

THEOREM 3. *Λ is continuous.*

PROOF. Since Λ is compact-valued, to show upper hemicontinuity it is sufficient to show that Λ has a closed graph. Suppose $\mu_n, \mu \in \mathscr{G}(S)$, $\mu_n \to \mu$, $\sigma_n \in \Lambda(\mu_n)$, and $\sigma_n \to \sigma$. We need to show that $\sigma \in \Lambda(\mu)$. First, $\sigma \in \mathscr{S}(S)$ since $\mathscr{S}(S)$ is closed.

Theorem 3.2 of Billingsley [2] implies that $\sigma|_{\mathscr{T}(S)} = \mu$ since $\sigma_n|_{\mathscr{T}(S)} = \mu_n$. If $(K, f, x, s) \in \operatorname{supp} \sigma$, then there exist $(K_n, f_n, x_n, s_n) \in \operatorname{supp} \sigma_n$ that converge to (K, f, x, s). Since $s_n \in K_n$, it follows that $s \in K$. Thus, $\sigma \in \Lambda(\mu)$.

Suppose $\mu \in \mathscr{G}(S)$, $\sigma \in \Lambda(\mu)$, and $\epsilon > 0$. Since Λ is compact-valued, to show lower hemicontinuity, it is sufficient to show that $\exists \delta > 0\ \forall \nu \in \mathscr{G}(S)$ s.t. $\rho(\mu, \nu) < \delta \exists \tau \in \Lambda(\nu)$ s.t. $\rho(\sigma, \tau) < 2\epsilon$. Because S is compact, there exists a partition $\{S_1, \dots, S_n\}$ of S satisfying $\operatorname{diam}(S_k) < \epsilon$ for $k = 1, \dots, n$. Let σ_k be defined by $\sigma_k(E \times R) = \sigma(E \times (R \cap S_k))/\sigma(\mathscr{T}(S) \times S_k)$ for $E \in \mathscr{F}(\mathscr{T}(S))$ and $R \in \mathscr{F}(S)$. Let $\mu_k = \sigma_k|_{\mathscr{T}(S)}$. Define a scalar multiplication $\odot$ by $(c \odot \mu)(E) = c\mu(\{(K, f, x): (K, f, cx) \in E\})$. Clearly, $\odot$ is continuous, and if $\mu \in \mathscr{G}(S)$, then $c \odot \mu$ satisfies all the properties of being a game (in particular, the nonnegative integer property) except that $(c \odot \mu)(\mathscr{T}(S)) = c$ instead of 1. By the continuity of the addition and scalar multiplication operators, $\exists \delta > 0\ \forall \nu \in \mathscr{G}(S)$ s.t. $d(\mu, \nu) < \delta \exists \nu_1, \dots, \nu_n \in \mathscr{G}(S)$, $c_1, \dots, c_n > 0$: $\nu = c_1 \odot \nu_1 + \cdots + c_n \odot \nu_n$, $\rho(\mu_k, \nu_k) < \epsilon$ and $\operatorname{supp} \nu_k \subseteq B_\epsilon(\operatorname{supp} \mu_k)$ for $k = 1, \dots, n$. Let (β, π) be a game with named players $(A, \mathscr{A}, \alpha)$ having the same distribution as ν_k (such a game exists by Example 2). Let $H: A \to \mathscr{K}(S)$ be defined by $H(a) = \beta(a) \cap \bar{B}_\epsilon(S_K)$. It is clear that H is nonempty-valued and lower measurable. Hence, by the Measurable Selection Theorem [12], $\exists \mathrm{s} \in \Lambda(\beta, \pi)\ \forall_\alpha a \in A$: $d(\mathrm{s}(a), S_k) \leqslant \epsilon$. Let $\tau_k = \alpha \circ (\beta \times \pi \times \alpha \times \mathrm{s})^{-1}$, and consider $\tau = c \odot \tau_1 + \cdots + c_n \odot \tau_n$, where $(c \odot \tau)(E) = c\tau(\{(K, f, x, s): (K, f, cx, s) \in E\})$. Clearly, $\tau \in \Lambda(\nu)$. Finally, we show that $\rho(\sigma, \tau) < 2\epsilon$. Suppose $E \in \mathscr{F}(\mathscr{T}(S))$ and $R \in \mathscr{F}(S)$. If $K = \{k: S_k \cap R \neq \varnothing\}$, then

$$\sigma(E \times R) \leqslant \sum_{k \in K} \sigma(E \times S_k) = \sum_{k \in K} c_k \sigma_k(E \times S) \leqslant \sum_{k \in K} c_k[\tau_k(E \times S) + \epsilon]$$

$$\leqslant \sum_{k \in K} \tau(E \times B_\epsilon(S_k)) + \epsilon \leqslant \tau(E \times B_{2\epsilon}(R)) + \epsilon < \tau(B_{2\epsilon}(E \times R)) + 2\epsilon.$$

If $K = \{k: B_\epsilon(S_k) \cap R \neq \varnothing\}$, then

$$\tau(E \times R) = \sum_{k \in K} c_k \tau_k(E \times R) \leqslant \sum_{k \in K} c_k[\sigma_k(E \times S_k) + \epsilon] < \sigma(B_{2\epsilon}(E \times R)) + 2\epsilon.$$

Even for finite player games, Nash equilibria do not exist in general. A convexity condition is often used to insure existence of Nash equilibria for finite player games. Theorem 4 generalizes results given by Schmeidler [15] and Mas-Colell [13]. In Theorem 5 we easily obtain the upper hemicontinuity of the Nash equilibrium correspondence.

DEFINITION. A game μ on a compact subset S of a locally convex linear topological space is *convex* if $\forall (K, f, x) \in \operatorname{supp} \mu\ \forall \xi \in \mathscr{M}(S)$: K is convex and $g(r) = f(r, (1 - x)\xi + xp(r))$ is quasiconcave in r.

THEOREM 4. *If μ is convex, then $\Phi(\mu)$ is nonempty.*

PROOF. This follows immediately from the Nash equilibrium existence theorem for games with named players found in Housman [9] and Theorem 1.

THEOREM 5. *Φ is compact-valued and upper hemicontinuous.*

PROOF. Since $\Phi(\mu) \subseteq \Lambda(\mu)$ and Λ is compact-valued, we need only show that Φ has a closed graph. Suppose μ_n, $\mu \in \mathscr{G}(S)$, $\sigma_n \to \sigma$, $\sigma_n \in \Phi(\mu_n)$, and $\sigma_n \to \sigma$. By Theorem 3, $\sigma \in \Lambda(\mu)$ and $\forall (K, f, x, s) \in \operatorname{supp} \sigma\ \forall r \in K$, $\exists (K_n, f_n, x_n, s_n) \in \operatorname{supp} \sigma_n \exists r_n \in K_n$: $(K_n, f_n, x_n, s_n) \to (K, f, x, s)$ and $r_n \to r$. Because $\sigma_n \in \Phi(\mu_n)$, it follows that $f_n(s_n, \sigma_n|_S) \geqslant f_n(r_n, \sigma_n|_S + x_n[p(r_n) - p(s_n)])$. Hence, by the various convergence and continuity properties of the entities involved, $f(s, \sigma|_S) \geqslant f(r, \sigma|_S + x[p(r) - p(s)])$.

While the Nash equilibrium correspondence is upper hemicontinuous, it is easy to provide counterexamples to lower hemicontinuity. This is also true for finite player games in the usual formulation. Nonetheless, there is a slightly weaker lower hemicontinuity type property that does hold for finite player games with named players.

DEFINITIONS. The feasible strategy profile σ is a γ-*Nash equilibrium* for the game μ if $\forall (K, f, x, s) \in \operatorname{supp} \sigma$ $\forall r \in K$: $f(s, \sigma|_S) \geqslant f(r, \sigma|_S + x[p(r) - p(s)]) - \gamma$. The set of all γ-Nash equilibria of the game μ is denoted by $\Phi_\gamma(\mu)$. The Nash equilibrium correspondence Φ is *nearly lower hemicontinuous* on G at μ if $\forall \gamma, \epsilon > 0$ $\exists \delta > 0$ $\forall \nu \in G$ s.t. $d(\mu, \nu) < \delta$: $\Phi(\mu) \subseteq B_\epsilon(\Phi_\gamma(\nu))$.

Housman [9] has shown that the Nash equilibrium correspondence for finite player games with named players is nearly lower hemicontinuous. This result does not hold for games without named players because of the possibility of "contaminating" any game with a very small game of matching pennies with pure strategies (see Theorem 6). Nonetheless, the Nash equilibrium correspondence is nearly lower semicontinuous on a restricted class of games (see Theorem 7).

THEOREM 6. *Φ is not nearly lower hemicontinuous on $(\mathscr{G})$ at any game μ for which $\Phi(\mu) \neq \varnothing$.*

PROOF. Let $A = \{1, 2\}$, $\alpha(1) = \alpha(2) = 1/2$, and $S = [-1, 1]$. Let (β, π) be defined by $\beta(a) = \{-1, 1\}$, $\pi(1, s) = -s(1)s(2)$, and $\pi(2, s) = s(1)s(2)$. Let $\nu = \alpha \circ (\beta \times \pi \times \alpha)^{-1}$. Clearly $\Phi_\gamma(\nu) = \varnothing$ for $\gamma < 2$. Let $\mu \in \mathscr{C}(S)$ and consider $\mu_c = (1 - c) \odot \mu + c \odot \nu$ where $\odot$ is as defined in the proof of Theorem 3. Again $\Phi_\gamma(\mu_c) = \varnothing$ for $\gamma < 2$ and $c > 0$, but $\mu_c \to \mu$ as $c \to 0$.

DEFINITION. The game μ is said to be equicontinuous if $\{f: (K, f, x) \in \operatorname{supp} \mu\}$ is a compact subset of $\mathscr{C}(S \times \mathscr{M}(S))$.

THEOREM 7. *Φ is nearly lower hemicontinuous on the space of convex games at an equicontinuous game.*

PROOF. Suppose μ is an equicontinuous game, $\sigma \in \Phi(\mu)$, and $\gamma, \epsilon > 0$. Since $\Phi(\mu)$ is compact, to show near lower hemicontinuity, it is sufficient to show that $\exists \delta > 0$ $\forall$ convex games ν s.t. $\rho(\mu, \nu) < \delta$ $\exists \tau \in \Phi_\gamma(\nu)$: $\rho(\sigma, \tau) < \epsilon$. Equicontinuity and the compactness of S and $\mathscr{M}(S)$ imply *uniform* equicontinuity: $\exists \epsilon' \in (0, \min\{\epsilon, \gamma/8\})$ $\forall (K, f, x) \in \operatorname{supp} \mu$ $\forall r, s \in S$ s.t. $d(r, s) < 2\epsilon'$ $\forall \xi, \eta \in \mathscr{M}(S)$ s.t. $\rho(\xi, \eta) < 4\epsilon'$: $|f(r, \xi) - f(s, \eta)| < \gamma/4$. The continuity of Λ implies that $\exists \delta \in (0, \epsilon')$ $\forall \nu \in \mathscr{G}(S)$ s.t. $\rho(\mu, \nu) < \delta$ $\exists \hat{\tau} \in \Lambda(\mu)$: $\rho(\sigma, \hat{\tau}) < \epsilon'$. Suppose ν is a convex game such that $\rho(\mu, \nu) < \delta$. By the above, $\exists \hat{\tau} \in \Lambda(\mu)$: $\rho(\sigma, \hat{\tau}) < \epsilon'$. If $\hat{\tau}$ were a γ-Nash equilibrium, we would be done; however, there can be a set of players who are not playing "γ-optimal" strategies. We change these players' strategies. Let $W = B_{\epsilon'}(\operatorname{supp} \sigma)$, $\tau_W(E) = \hat{\tau}(E \cap W)$, and $c = 1 - \tau_W(W)$. Consider the game ν' defined by $\nu'(E) = \nu(\{(K, f, x)\colon (K, g, c^{-1}x) \in E$ where $g(s, \xi) = f(s, c\xi + \tau_W|_S)\})$. In words, ν' is the game played by the players of ν whose strategy choice (as a part of τ') was not in W when the players with strategy choices in W do not change their strategy choice. Because ν is a convex game, ν' is a convex game. So, Theorem 4 implies that $\exists \tau' \in \Phi(\nu')$. Let $\tau = \tau_W + c \odot \tau'$ where $\odot$ is the scalar multiplication defined in the proof of Theorem 3. Clearly, $\tau \in \Lambda(\nu)$ and $\rho(\sigma, \tau) < 2\epsilon'$. Suppose $(L, g, y, v) \in \operatorname{supp} \tau$ and $u \in L$. If $(L, g, y, v) \notin \operatorname{supp} \tau_W$, then $g(u, \tau|_S) \geqslant g(v, \tau|_S + y[p(u) - p(v)])$ because $\tau' \in \Phi(\nu')$. Suppose now that $(L, g, y, v) \in \operatorname{supp} \tau_W$ and $u \in L$. Then $\exists (K, f, x, s) \in \operatorname{supp} \sigma \exists r \in K$: $d(g, f) < 2\epsilon'$, $d(y, x) < 2\epsilon'$, $d(v, s) < 2\epsilon'$, and $d(u, r) < 2\epsilon'$. Also, note that if $\xi = \sigma|_S + x[p(r) - p(s)]$ and $\eta = \tau|_S +$

$y[p(u) - p(v)]$, then $\rho(\xi, \eta) < 4\epsilon'$. Hence,

$$\begin{aligned} g(\mu, \tau|_S) - g(v, \eta) \geq & -|g(u, \tau|_S) - f(u, \tau|_S)| \\ & -|f(u, \tau|_S) - f(s, \sigma|_S)| \\ & +[f(s, \sigma|_S) - f(r, \xi)] \\ & -|f(r, \xi) - f(v, \eta)| \\ & -|f(v, \eta) - g(v, \eta)| \\ \geq & -2\epsilon' - \gamma/4 + 0 - \gamma/4 - 2\epsilon' \geq \gamma. \end{aligned}$$

Therefore, $\tau \in \Phi_\gamma(\nu)$.

REMARK 7. If S is a complete metric space with a bounded metric, then Theorem 7 must be modified to consider a *uniformly* equicontinuous game rather than just an equicontinuous game. Example 6 of Housman [9] shows that the theorem will not be true, in general, if the equicontinuity assumption is relaxed.

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References

[1] Aumann, R. J. (1964). Markets with a Continuum of Players. *Econometrica* **32** 39–50.
[2] Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley and Sons, New York.
[3] Debreu, G. (1966). Integration of Correspondences. *Proc. Fifth Berkeley Sympos. Statistics and Probability*. Vol. II. Part 1. Univ. of California Press, Berkeley.
[4] Dubey, P. and Shapley, L. S. (1977). Noncooperative Exchange with a Continuum of Traders. Cowles Foundation Discussion Paper No. 447.
[5] Dunford, N. and Schwartz, J. T. (1958). *Linear Operators. Part I.* Interscience Publishers, New York.
[6] Green, E. (1984). Continuum and Finite-Player Noncooperative Models of Competition. *Econometrica* **4** 975–993.
[7] Hart, S., Hildenbrand, W. and Kohlberg, E. (1974). On Equilibrium Allocations as Distributions on the Commodity Space. *J. Math. Economics* **1** 159–166.
[8] Housman, D. (1983). Some Noncooperative Game Models of Exchange. Ph.D. Dissertation, Cornell University.
[9] ——. (1986). The Continuity of the Nash Equilibrium Correspondence. Manuscript.
[10] Kannai, Y. (1970). Continuity Properties of the Core of a Market. *Econometrica* **38** 791–815.
[11] —— and Peleg, B. (1979). An Approach to the Problem of Efficient Distribution of the Labor Force. *Applied Game Theory*. Physica-Verlag, Wuerzburg, Germany.
[12] Kuratowski, K. and Ryll-Nardzewski, C. (1965). A General Theorem on Selectors. *Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys.* **13** 397–403.
[13] Mas-Colell, A. (1984). On a Theorem of Schmeidler, *J. Math. Economics* **13** 201–206.
[14] Nti, K. (1988). Capital Asset Prices in an Oligopolistic Market. *J. Economics (Z. Nationalökonomie)* (to appear).
[15] Schmeidler, D. (1973). Equilibrium Points of Nonatomic Games. *J. Statistical Physics* **7** 295–300.
[16] Varadarajan, V. S. (1965). Measures on Topological Spaces. *Amer. Math. Soc. Transl. Ser. 2.* **48** 161–228.
[17] Willard, S. (1970). *General Topology*. Addison-Wesley, Reading, MA.

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