# A Game Theory Path To Quantitative Literacy

## Deterministic Games

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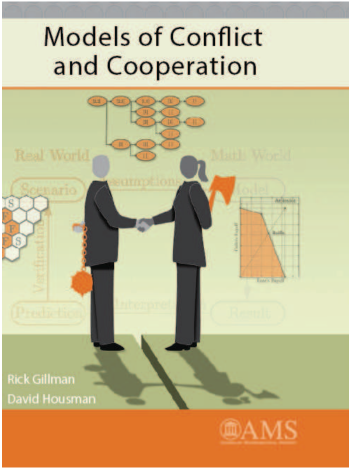
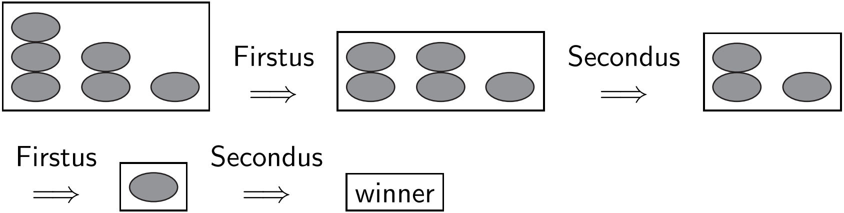
# Overview




- Game Types
  - Deterministic (Nim)
  - Strategic (Mating)
  - Bargaining (Acme Industrial)
  - Coalition (EPA)
- Workshop Approach
  - Classroom experience
  - Learning objectives and pedagogy discussion
  - Materials
  - www.goshen.edu/dhousman and scroll to A Game Theory Path To Quantitative Literacy

# Nim

**Nim** is played with several heaps of beans (or beads or tiles). The two players, Firstus and Secondus, take turns moving, with Firstus making the first move. When it is her or his turn to move, the player removes one or more beans from a single heap. The player who removes the last bean is the winner.




Play and share your observations!

# Heuristics versus Strategies

**Definition**

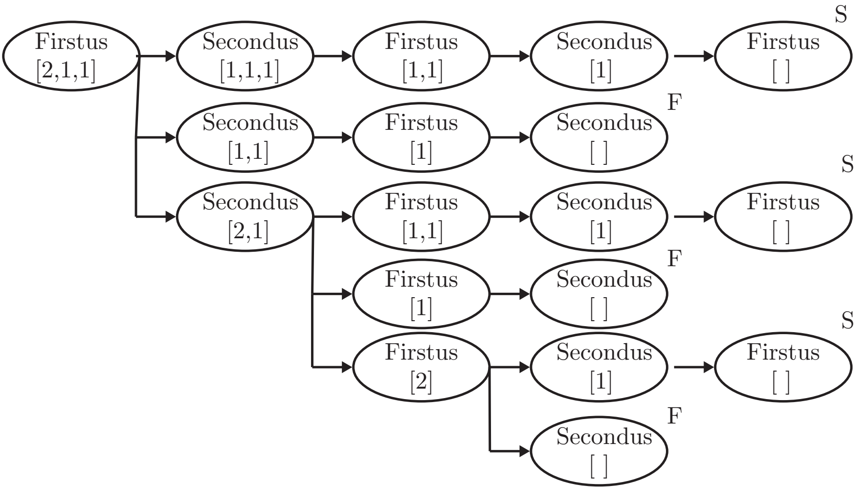
A *heuristic* is a guide for making decisions.

**Definition**

A *strategy* is a complete and unambiguous description of what to do in every possible situation.

- Even the heaps.
- Mirror the other player.
- Remove one bean from a smallest heap.

# Game Tree

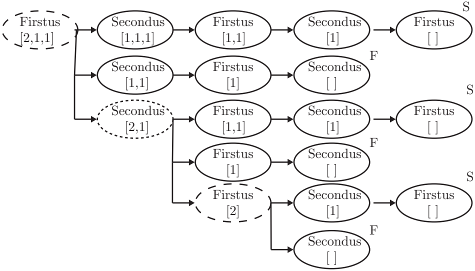


Firstus [2,1,1]
Secondus [1,1,1]
Firstus [1,1]
Secondus [1]
Firstus [ ]
S
Secondus [1,1]
Firstus [1]
Secondus [ ]
F
Secondus [2,1]
Firstus [1,1]
Secondus [1]
Firstus [ ]
S
Firstus [1]
Secondus [ ]
F
Firstus [2]
Secondus [1]
Firstus [ ]
S
Secondus [ ]
F

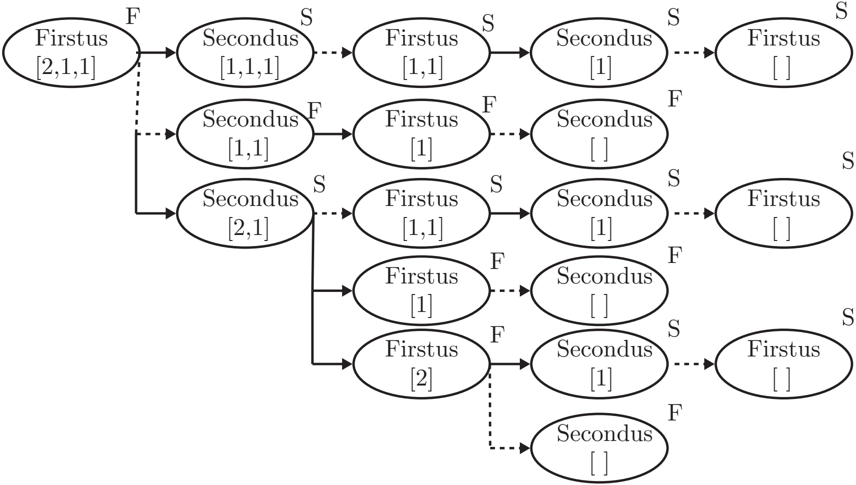
# Game Tree Strategies

Firstus' strategies:

1. Remove 1 bean from the 2 bean heap.
2. Remove 2 beans from the 2 bean heap.
3. Remove 1 bean from a 1 bean heap and if faced with [2], remove 1 bean.
4. Remove 1 bean from a 1 bean heap and if faced with [2], remove 2 beans.



# Backward Induction



# Nim-Sum Strategy

## Definition

The *nim-sum strategy* is to remove beans from a heap so that the nim-sum of the resulting game is zero.

$$\begin{array}{ccccc} 6 & = & 1 & 1 & 0 \\ 4 & = & 1 & 0 & 0 \\ 4 & = & 1 & 0 & 0 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 1 & 1 & 1 \end{array} \qquad \begin{array}{ccccc} 6 & = & 1 & 1 & 0 \\ 4 & = & 1 & 0 & 0 \\ 3 & = & 0 & 1 & 1 \\ 1 & = & 0 & 0 & 1 \\ \hline & & 0 & 0 & 0 \end{array}$$

# Proof that Nim-Sum is a Winning Strategy

If the game has a non-zero nim-sum, there must be at least one column whose nim-sum is 1. Consider the left-most column whose nim-sum is currently 1; call it column $c$. There must be a row, call it row $r$, having a 1 in column $c$ (for otherwise, the nim-sum of column $c$ would have been 0). For each column with nim-sum 1 that row $r$ contributes to by having a 1 in it, remove sufficient beans to convert the 1 to 0. For each column with nim-sum 1 that row $r$ does not contribute to by having a 0 in it, add sufficient beans to convert the 0 to 1. This results in a net removal of beans because removing the beans necessary to change the 1 in column $c$ to 0 will remove more beans than will need to be returned for those columns that a 0 needs to be changed to a 1. Hence, the next position can be made zero nim-sum.

If the game has a zero nim-sum, then the removal of beans from a heap will change at least one 1 to a 0 in the corresponding row. This will leave an odd number of 1s in its column, and therefore a sum of 1 in that column. Hence, the next position must be non-zero nim-sum.

# Learning Objectives

- Actively process information
- Use precise language
- Develop number sense
- Use systematic thinking

# Homework

- Play and analyze Poison
- Play and analyze Hex
- Create and analyze variations of Nim, Poison, and Hex

# Homework One

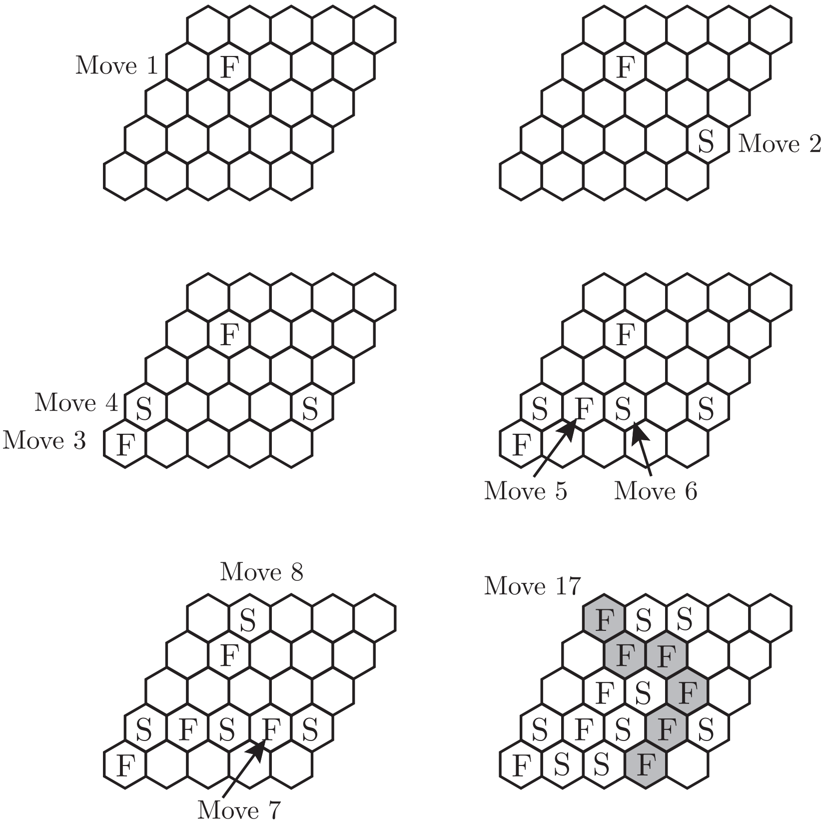
- Play and analyze **Poison**
- Play and analyze **Hex**
- Create and analyze variations of **Nim**, **Poison**, and **Hex**

### Poison

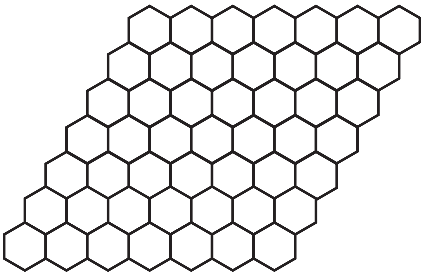
In the game **Poison**, each of two players take turns selecting one or two tiles from a single pile of tiles. The person who is forced to pick up the last tile loses. For example, if there are 4 tiles in the pile initially, Firstus might select one tile leaving three on the pile. Secondus may then take two tiles leaving one tile on the pile. Firstus would then be forced to take the last tile resulting in Secondus being declared the winner.

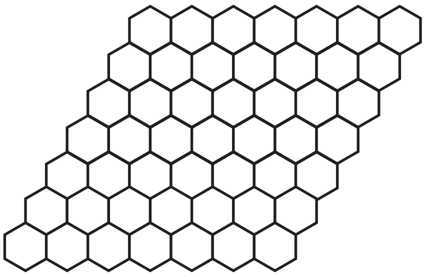
### Hex

The game of **Hex** is played on a rhombus shaped grid of hexagons, in which two hexagons are considered adjacent if they share a side. Firstus and Secondus take turns moving. When it is her or his turn to move, the player captures one hexagon by writing her or his initial in the hexagon. They may capture any hexagon that has not already been captured, whether or not it is adjacent to a previously captured hexagon. Firstus wins if she captures a path of hexagons from top to bottom of the rhombus. Secondus wins if he captures a path of hexagons from left to right of the rhombus. The first eight moves and the winning move of a $5 \times 5$ **Hex** game are shown below.




Here are some blank HEX boards to play with:





# A Game Theory Path To Quantitative Literacy

## Strategic Games

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# Gas Stations Scenario

If both of the gas stations in a small community charge the same prices for gasoline, then they will each sell roughly the same amount of gasoline and obtain a certain profit. If RG Oil sets its price slightly lower than DH Petrol, then RG Oil will attract customers away from DH Petrol. Although RG Oil will obtain slightly less profit from each customer (because of its lower prices), it will attract so many additional customers (again because of its lower prices) that its overall profit will increase. Of course, once RG Oil lowers its prices, DH Petrol loses customers resulting in less profit. In this situation, DH Petrol can increase its profits only by also lowering its prices. Of course, if both gas station choose to lower their prices, then no one gains market share and each receives a smaller profit. If both gas stations could together choose a high price, then each will make a larger profit with the same market share. Of course, with its competitor choosing a high price, either gas station can make even more profit by choosing a lower price.

# Gas Stations Model

| Gas Stations Outcomes | | DH Petrol CUT PRICE | HOLD PRICE |
|---|---|---|---|
| RG Oil | CUT PRICE | Same market shares, less revenue. | RG gains revenue, DH loses revenue. |
| | HOLD PRICE | RG loses revenue, DH gains revenue. | Status Quo. |

Since gas stations try to maximize revenue, the ordinal payoff matrix is

| Gas Stations Ordinal Payoffs | | DH Petrol CUT PRICE | HOLD PRICE |
|---|---|---|---|
| RG Oil | CUT PRICE | $(2, 2)$ | $(4, 1)$ |
| | HOLD PRICE | $(1, 4)$ | $(3, 3)$ |

Play the Gas Stations Game!

# Gas Stations Analysis: Prudential Strategy

## Definition

A *prudential strategy* for a player is a strategy, from among the player's available strategies, that will maximize the minimum payoff the player could receive when choosing that strategy.payoff matrix.

| Gas Stations Ordinal Payoffs | | DH Petrol CUT PRICE | HOLD PRICE | Minimum RG Oil |
|---|---|---|---|---|
| RG Oil | CUT PRICE | $(2, 2)$ | $(4, 1)$ | 2 |
| | HOLD PRICE | $(1, 4)$ | $(3, 3)$ | 1 |
| Minimum | DH Petrol | 2 | 1 | |

RG Prudential: CUT.
DH Prudential: CUT.

# Gas Stations Analysis: Dominating Strategy

## Definition

Strategy $X$ *dominates* strategy $Y$ if the payoff associated with playing strategy $X$ is larger than the payoff associated with playing strategy $Y$ in every situation. Strategy $X$ is said to be *dominating* and strategy $Y$ is said to be *dominated*.

| Gas Stations Ordinal Payoffs | | DH Petrol CUT PRICE | HOLD PRICE |
|---|---|---|---|
| RG Oil | CUT PRICE | $(2, 2)$ | $(4, 1)$ |
| | HOLD PRICE | $(1, 4)$ | $(3, 3)$ |

RG: CUT dominates HOLD.
DH: CUT dominates HOLD.

# Gas Stations Analysis: Nash Equilibrium

## Definition

A *Nash equilibrium* is an ordered set of strategy choices, one for each player, for which by unilaterally changing her or his strategy, no player can improve her or his own payoff.

| Gas Stations Ordinal Payoffs | | DH Petrol CUT PRICE | HOLD PRICE |
|---|---|---|---|
| RG Oil | CUT PRICE | (2, 2) | (4, 1) |
| | HOLD PRICE | (1, 4) | (3, 3) |

Nash Equilibrium: (CUT, CUT)

# Gas Stations Analysis: Efficient Strategy Pair

**Definition**

An *efficient strategy pair* is an ordered set of strategy choices, one for each player, for which by changing their strategies together, one player can increase his or her payoff only by decreasing another player's payoff.

| Gas Stations Ordinal Payoffs | | DH Petrol CUT PRICE | HOLD PRICE |
|---|---|---|---|
| RG Oil | CUT PRICE | $(2,2)$ | $(4,1)$ |
| | HOLD PRICE | $(1,4)$ | $(3,3)$ |

Efficient Strategy Pairs:
(HOLD, CUT), (HOLD, HOLD), and (CUT, HOLD).

# Mating Scenario

A common biological component of mating is the problem of a female of a given species trying to get a male to stay around and help raise a family of babies, instead of going off and propagating his genes elsewhere. One possible technique for doing this is to insist on a long and arduous courtship before mating. Suppose a female can be either Coy (insist on courtship) or Fast (be willing to mate with anyone), and a male can be either Faithful (go through a courtship and then help raise the babies) or Philandering (be unwilling to go through a courtship and desert any female after mating). Suppose the payoff to each parent of babies is $+15$, and the total cost of raising babies is $-20$, which can be split equally between both parents, or fall entirely on the female if the male deserts. Suppose the cost of a long courtship is $-3$ to each player.

Model as a strategic game and analyze!

# Mating Game

| Mating Outcomes | | Male FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | Courtship yielding a child raised by both parents. | No courtship and no child. |
| | FAST | A child raised by both parents. | A child raised by the female alone. |

Since courtship translates into a loss of 3 points for each individual and a child translates into a gain of 15 points for each parent and a total loss of 20 by the parent(s) who raise the child, then the cardinal payoff matrix is ...

| Mating Cardinal Payoffs | | Male FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | $(2,2)$ | $(0,0)$ |
| | FAST | $(5,5)$ | $(-5,15)$ |

Play the Mating Game!

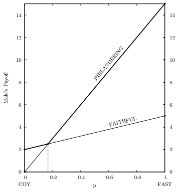
# Mating Game Analysis

| Mating<br>Cardinal Payoffs | | Male<br>FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | (2, [2]) | ([0], 0) |
| | FAST | ([5], 5) | (−5, [15]) |

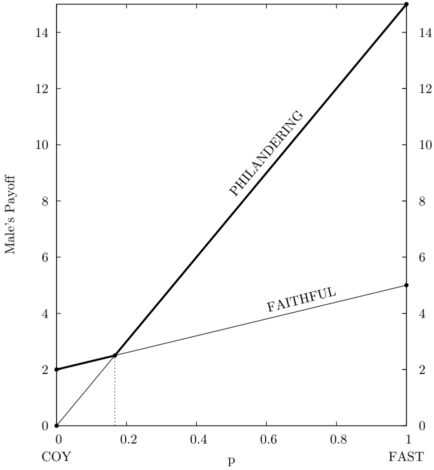
- COY is Female prudential; FAITHFUL is Male prudential
- No strategy dominates another strategy
- No (pure strategy) Nash equilibrium
- (FAST, FAITHFUL) and (FAST, PHILANDERING) are the efficient strategy pairs

What if the male faces a female using a **mixed** strategy?

| Mating<br>Male Cardinal Payoffs | | Male<br>FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | 2 | 0 |
| | FAST | 5 | 15 |
| $(1-p)\text{COY} + p\text{FAST}$ | | $2(1-p)+5p$ | $0(1-p)+15p$ |



# Mating Game Analysis




Female strategy:

$$(1-p)\text{COY} + p\text{FAST}$$

Male FAITHFUL payoff:

$$2(1-p) + 5p$$

Male PHILANDERING payoff:

$$0(1-p) + 15p$$

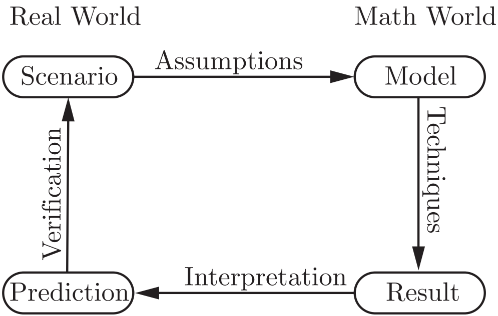
When $p = \frac{1}{6}$, any strategy for the male is a best response.

Nash equilibrium:

$$\left(\frac{5}{6}\text{COY} + \frac{1}{6}\text{FAST}, \frac{5}{8}\text{FAITH} + \frac{3}{8}\text{PHIL}\right)$$

# Learning Objectives

- Use the mathematical modeling process




- Use algorithmic thinking (vs. formula)

# Homework

- Model and analyze the **Matches** scenario.
- Model and analyze the **Soccer Penalty Kicks** scenario.

# Homework Two

- Model and analyze the **Matches** scenario.
- Model and analyze the **Soccer Penalty Kicks** scenario.

### Matches

Rose and Colin just met and had a pleasant conversation at a local coffee house. Just before they depart, Rose says, “Hope to see you at the match tomorrow,” and Colin responds, “It’s a date!” As each heads off for ”the match” the next day, each suddenly realizes that there are actually two big matches scheduled that day: tennis and soccer. From their conversation, each knows that Rose likes tennis and Colin likes soccer. Unfortunately, neither one had gotten the other’s last name or telephone number, and the two games are being held at fields about one hour apart.

### Soccer Penalty Kicks

In professional soccer, essentially no time passes between a penalty kicker’s kick to the right or left of the goal and the goalie’s decision on which corner to defend. These decisions can be assumed to be made simultaneously. Suppose the kicker’s accuracy and speed is greater when kicking to the left side of the goal. Specifically, the kicker has a 100% chance of scoring an undefended left side, a 80% chance of scoring an undefended right side, a 10% chance of scoring a defended left side, and a 0% chance of scoring a defended right side.

# A Game Theory Path To Quantitative Literacy

## Bargaining Games

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# Overview

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# Acme Industrial Scenario

1. Wages. Labor wants a 3% annual increase and Management wants no annual increase.
2. Benefits. Labor wants to maintain all current benefits and Management wants employees to pay for one-half of their medical insurance.
3. Security. Labor wants assurances that no jobs will be eliminated during the contract period and Management wants the capability to eliminate up to six hundred union jobs during the contract period.

Assigning cardinal payoffs:

| Issue | Labor Worth | Man Worth |
|---|---|---|
| Wages | 50 | 10 |
| Benefits | 30 | 30 |
| Security | 20 | 60 |
| Total | 100 | 100 |

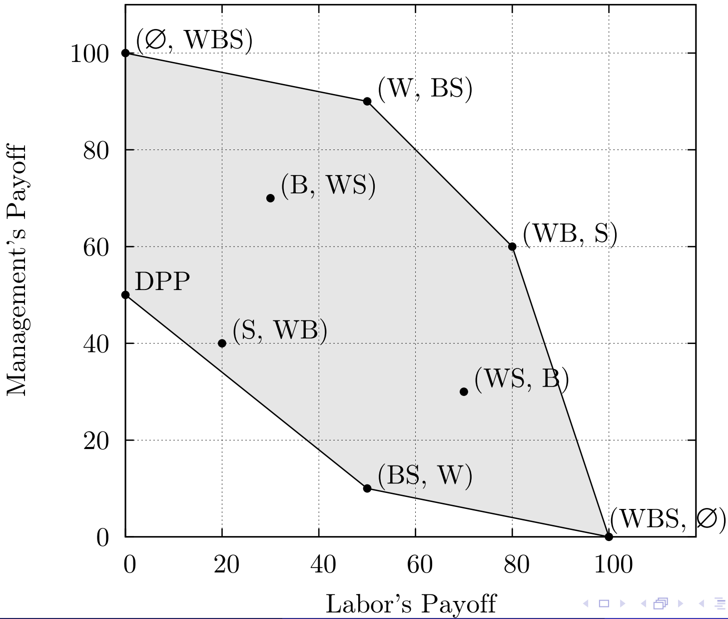
What happens if there is no agreement? Strike? Lockout? Bankruptcy? Mandated arbitration?
Labor worth: 0
Management worth: 50

# Acme Industrial Scenario

| Issue | Labor | Manage |
|---|---|---|
| Wages | 50 | 10 |
| Benefits | 30 | 30 |
| Security | 20 | 60 |
| Total | 100 | 100 |

| Outcome | Issue Winner | | | Labor | Manage |
|---|---|---|---|---|---|
| Name | Wages | Benefits | Security | Payoff | Payoff |
| (WBS, ∅) | L | L | L | 100 | 0 |
| (WB, S) | L | L | M | 80 | 60 |
| (WS, B) | L | M | L | 70 | 30 |
| (W, BS) | L | M | M | 50 | 90 |
| (BS, W) | M | L | L | 50 | 10 |
| (B, WS) | M | L | M | 30 | 70 |
| (S, WB) | M | M | L | 20 | 40 |
| (∅, WBS) | M | M | M | 0 | 100 |

# Acme Industrial Game



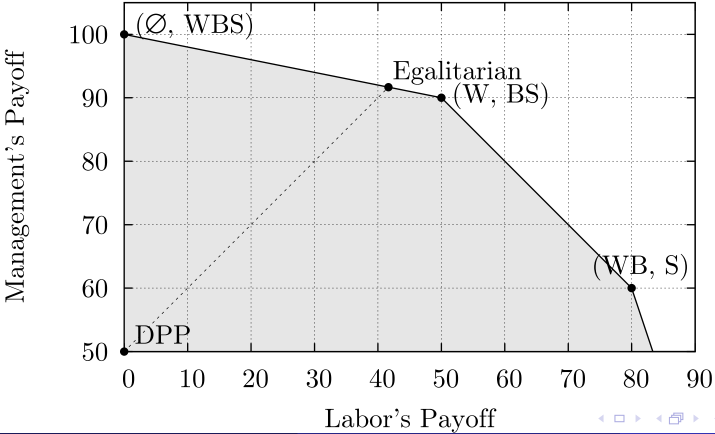

| I | Lab | Man |
|---|---|---|
| W | 50 | 10 |
| B | 30 | 30 |
| S | 20 | 60 |
| | | |
| D | 0 | 50 |

Play and share your observa-tions!

# Egalitarian Method

**Definition**

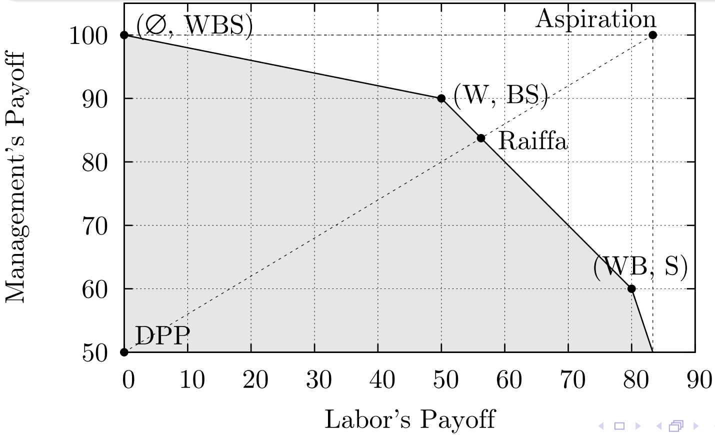
First give the players their disagreement payoff pair $(r, c)$. Then give the same additional amount, $t$, to each player, with the value $t$ chosen as large as possible so that the resulting egalitarian payoff pair $(r + t, c + t)$ remains feasible.



# Raiffa Method

## Definition

Find the aspiration payoff pair formed by the maximum individual payoffs among the rational payoff pairs. The Raiffa payoff pair is the efficient payoff pair on the line segment formed between the disagreement and aspiration payoff pairs.



# Comparison

| Property | Method | | |
|---|---|---|---|
| | Egalitarian | Raiffa | Nash |
| Efficient | Yes | Yes | Yes |
| Unbiased | Yes | Yes | Yes |
| Rational | Yes | Yes | Yes |
| Scale Invariant | No | Yes | Yes |
| Strongly Monotone | Yes | No | No |
| Individually Monotone | Yes | Yes | No |
| Independent of Irrelevant Payoff Pairs | Yes | No | Yes |

# Learning Objectives

- Realize connections between geometry and algebra
- Use algebraic methods
- Use properties to compare methods

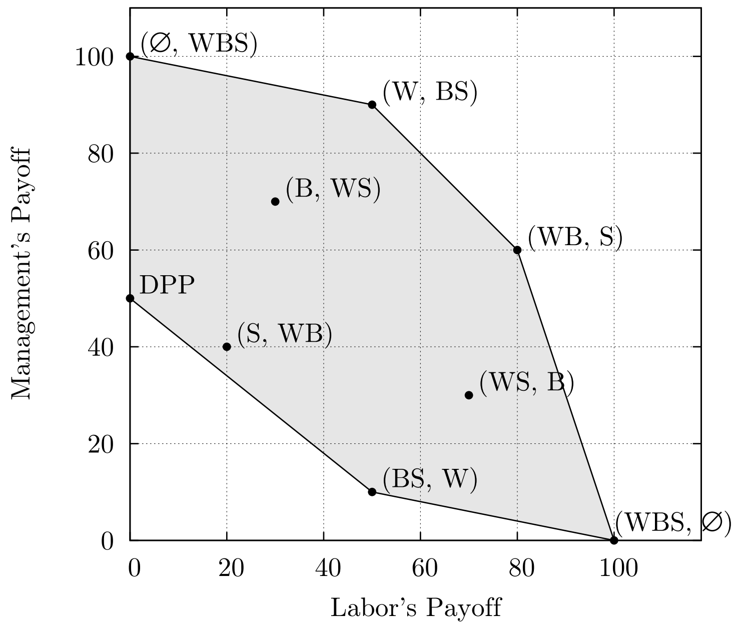
# Homework

- Read the document online describing the Nash method, and use it to analyze **Acme Industrial**.
- Model the **Mating** scenario as a bargaining game and complete the analysis of the game.

# Homework Three

- Read the document on line describing the Nash method, and use it to analyze **Acme Industrial**.
- Model the **Mating** scenario as a bargaining game and complete the analysis of the game.

**Acme Industrial**




**Mating Scenario**

A common biological component of mating is the problem of a female of a given species trying to get a male to stay around and help raise a family of babies, instead of going off and propagating his genes elsewhere. One possible technique for doing this is to insist on a long and arduous courtship before mating. Suppose a female can be either Coy (insist on courtship) or Fast (be willing to mate with anyone), and a male can be either Faithful (go through a courtship and then help raise the babies) or Philandering (be unwilling to go through a courtship and desert any female after mating). Suppose the payoff to each parent of babies is +15, and the total cost of raising babies is -20, which can be split equally between both parents, or fall entirely on the female if the male deserts. Suppose the cost of a long courtship is -3 to each player. What is the best decision for a female, or male, to make in this scenario?

| Mating Cardinal Payoffs | | Male FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | $(2, 2)$ | $(0, 0)$ |
| | FAST | $(5, 5)$ | $(-5, 15)$ |

# Nash method

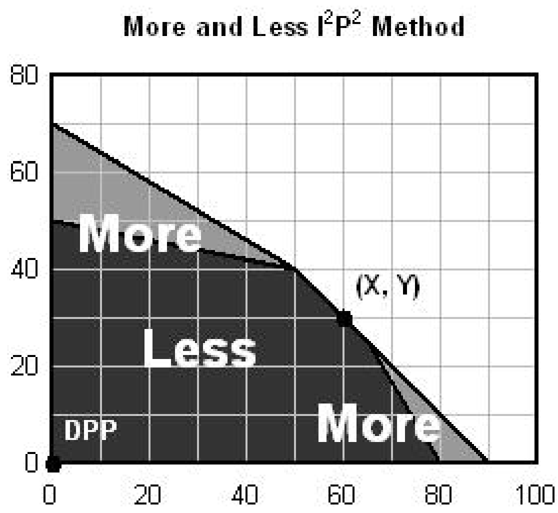
The egalitarian method has been recommended as a fair approach when player payoffs are measured by comparable scales such as money or probability of winning a prize. The Raiffa method has been recommended as a fair approach when, as is typically assumed, player payoffs cannot be directly compared. Both solutions are efficient and unbiased. The egalitarian method is then characterized by adding strongly monotone to the list of desired fairness properties. Alternatively, the Raiffa method is characterized by adding scale invariant and individually monotone to the list of desired fairness properties. In this section, we introduce a third monotonicity property and compare it to the previous two.

Returning to a previously described analogy, suppose that Rose is getting hungry at the basketball game. In the past, the concession stand has offered hot dogs, hamburgers, and chicken tenders. Suppose that chicken tenders is Rose's most preferred option. When Rose reaches the concession stand, she finds out that pulled pork sandwiches have been added to the menu. Rose may still order the chicken tenders, but she may order the pulled pork sandwich if she prefers it to the chicken tenders. The addition of more options has left Rose at least as well off as before (and perhaps better). This is the gist of the strongly and individually monotone properties for bargaining game solutions: the addition of feasible payoff pairs should leave both players or one player, respectively, at least as well off as before.

When John Nash introduced bargaining games in 1950, he had a different view of monotonicity. Suppose when Rose reaches the concession stand, she finds out that only hamburgers and chicken tenders are available. Since Rose had planned to choose chicken tenders when she thought hot dogs, hamburgers, and chicken tenders would be offered, she should certainly still choose chicken tenders when hot dogs are eliminated from consideration. In a sense, the hot dog option was irrelevant to Rose's decision between chicken tenders and hamburgers. He formalized this notion of monotonicity with the following definition.

**Independent of Irrelevant Payoff Pairs ($I^2P^2$)** Given a bargaining game, an *independent of irrelevant payoff pairs* method produces the same payoff pair $(X, Y)$ in any new game as it produced in the original game, as long as the new game was obtained by eliminating some feasible payoff pairs from the original game other than the payoff pair $(X, Y)$ produced for the original game.

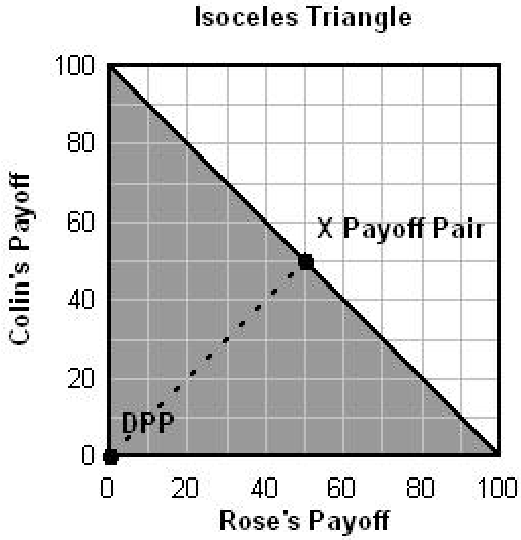
For example, consider **More**, whose rational payoff pairs consist of both the light and dark grey regions in the graph below, and **Less**, whose rational payoff pairs consist of only the dark grey region. Notice that **Less** can be obtained from **More** by eliminating the light grey regions of rational payoff pairs.




If a method is $I^2P^2$ and produces $(X, Y)$ for **More,** then that method must also produce $(X, Y)$ for **Less**.

Suppose that the X method is efficient, unbiased, scale invariant, and $I^2P^2$. Then, on the game **Isoceles**

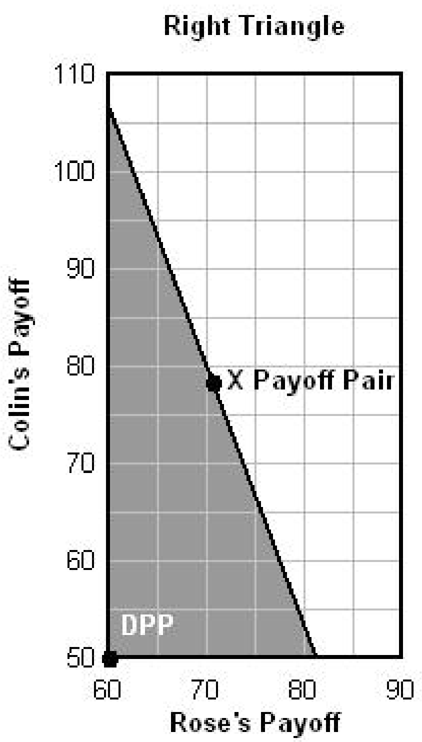
**Triangle**, illustrated below, which is symmetric, the X method must produce the payoff pair $(50, 50)$.




We can now use the rescaling

$$
\begin{aligned}
x' &= 60 + \frac{81.25 - 60}{100}x \\
y' &= 50 + \frac{106.67 - 50}{100}x
\end{aligned}
$$

to transform pairs $(x, y)$ in **Isoceles Triangle** into pairs $(x', y')$ in the game **Right Triangle**, shown below.




The three **Isoceles Triangle** vertices $(0, 0)$, $(100, 0)$, and $(0, 100)$ are transformed into the **Right Triangle** vertices $(60, 50)$, $(81.25, 50)$, and $(60, 106.67)$, respectively. Further, the **Isoceles Triangle** payoff pair

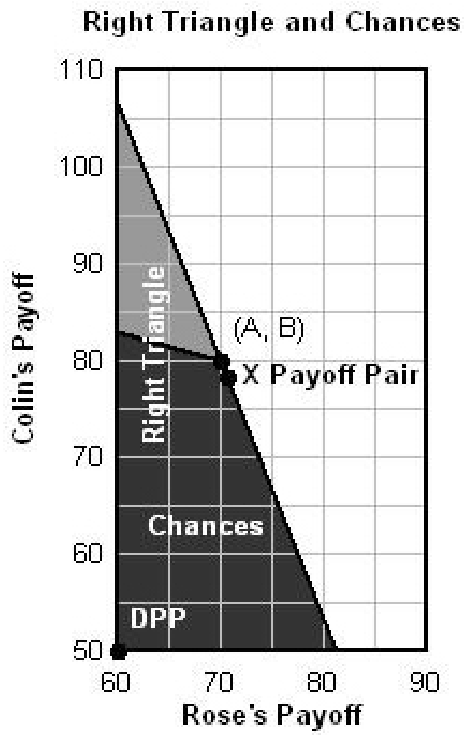
$(50, 50)$ is transformed into the **Right Triangle** payoff pair $(70.6, 78.3)$**.** This is true in general: midpoints of line segments are transformed to midpoints of line segments under linear transformations. Further, since the X method is scale invariant and produces $(50, 50)$ for **Isoceles Triangle**, it must also produce $(70.6, 78.3)$ as the payoff pair when applied to **Right Triangle**. In general, the X method must always produce the midpoint of the hypotenuse when the region of rational payoff pairs is a right triangle.

Now lets consider the bargaining game **Chances**, which has the payoff matrix given here:

| Chances | | Colin | |
|---|---|---|---|
| (Rose, Colin) | | A | B |
| Rose | A | $(50, 20)$ | $(70, 80)$ |
| | B | $(0, 100)$ | $(100, 0)$ |

We assume that the players will use their prudential strategy if they do not come to an agreement, resulting in the disagreement payoff pair $(60, 50)$.

Observe, coincidentally, that **Chances** can be obtained from **Right Triangle** by eliminating the feasible payoff pairs in the region shaded light grey, as shown here.




The X method is $I^2P^2$and produces $(70.6, 78.3)$ as a payoff pair for **Right Triangle.** Since $(70.6, 78.3)$ is feasible in **Chances**, the X method must produce $(70.6, 78.3)$ as a payoff pair for **Chances** as well. Thus, any efficient, unbiased, scale invariant, and $I^2P^2$ method applied to **Chances** must produce $(70.6, 78.3)$.

In summary, if the region of rational payoff pairs is a right triangle, an efficient, unbiased, and scale invariant method must produce the midpoint of the hypotenuse. If the region of rational payoff pairs is not a right triangle but we are able to extend the region to a right triangle whose hypotenuse midpoint is still in the original region of rational payoff pairs, then an $I^2P^2$ method will produce this midpoint. John Nash created an algorithm to find such a triangle:

**Nash Method** Extend one of the efficient and rational line segments to the disagreement payoff lines forming the hypotenuse to a right triangle. If the midpoint of the hypotenuse is rational, then it is the *Nash payoff pair*. If the midpoint is to the left or right of the rational payoff pairs on the hypotenuse, then select the next efficient and rational line segment in the corresponding direction and repeat these steps from the beginning. If at some stage you are supposed to reconsider a line segment, then the payoff pair shared by the two most recently considered efficient and rational line segments is the *Nash payoff pair*.

# A Game Theory Path To Quantitative Literacy

## Coalition Games

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January 2014

# EPA

The Environmental Protection Agency (EPA) has mandated improvements in the sewage treatment facilities in the cities of Avon, Barport, Claron, and Delmont. Each city could work separately, but $140 million would be saved by all four working together. Some smaller groups of cities also can save money as indicated in the table below.

| Coalition | ABCD | ABC | ABD | ACD | AB | others |
|---|---|---|---|---|---|---|
| Savings | 140 | 108 | 96 | 84 | 24 | 0 |

For cooperation to occur and the savings to be obtained, there must be a signed agreement among the cities stating how the savings is to be allocated among the cities. With no agreement, each city saves nothing ($0 million).

Play the game and share observations!

# A Very Simple Game

| Very Simple | |
|---|---|
| Coalition | Gain |
| ABCD | 100 |
| other | 0 |

What is the fair allocation in this game?

Fairness dictates that each player should receive 25.

## Definition

An allocation is *efficient* if it is impossible to increase the payoff of one player without decreasing the payoff to another player.

## Definition

An allocation is *unbiased* if players who are distinguishable only by their names are allocated the same payoff.

# A Somewhat Simple Game

| Somewhat Simple | |
|---|---|
| Coalition | Gain |
| ABCD | 150 |
| ABC | 150 |
| other | 0 |

What is the fair allocation in this game?

Fairness dictates that D should receive 0 and each other player should receive 50.

## Definition

An allocation is *subsidy free* if players that never contribute to or detract from gains are allocated zero.

# A Not So Simple Game

| Coalition | Gain |
|---|---|
| ABCD | 250 |
| ABC | 150 |
| other | 0 |

What is the fair allocation in this game?

Unbiased implies that players A, B, and C should receive the same amount, but it is unclear how much D should receive.

# A Not So Simple Game

| Coalition | Very Simple Gain | | Somewhat Simple Gain | | Not So Simple Gain |
|---|---|---|---|---|---|
| ABCD | 100 | $+$ | 150 | $=$ | 250 |
| ABC | 0 | $+$ | 150 | $=$ | 150 |

| Player | Very Simple Payoff | | Somewhat Simple Payoff | | Not So Simple Payoff |
|---|---|---|---|---|---|
| A, B or C | 25 | $+$ | 50 | $=$ | 75 |
| D | 25 | $+$ | 0 | $=$ | 25 |

**Definition**

An allocation method for coalition games is *additive* if whenever a coalition game is the sum or difference of other coalition games, the allocation for the original coalition game is the corresponding sum or difference of the allocations of the other coalition games.

# Properties Applied to the EPA Game

| Coalition | G1 | | G2 | | G3 | | G4 | | G5 | | EPA |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ABCD | 24 | + | 84 | + | 72 | + | 84 | − | 124 | = | 140 |
| ABC | 24 | + | 0 | + | 0 | + | 84 | − | 0 | = | 108 |
| ABD | 24 | + | 0 | + | 72 | + | 0 | − | 0 | = | 96 |
| ACD | 0 | + | 84 | + | 0 | + | 0 | − | 0 | = | 84 |
| AB | 24 | + | 0 | + | 0 | + | 0 | − | 0 | = | 24 |
| anything else | 0 | + | 0 | + | 0 | + | 0 | − | 0 | = | 0 |
| **Player** | **A1** | | **A2** | | **A3** | | **A4** | | **A5** | | **EPA** |
| A | 12 | + | 28 | + | 24 | + | 28 | − | 31 | = | 61 |
| B | 12 | + | 0 | + | 24 | + | 28 | − | 31 | = | 33 |
| C | 0 | + | 28 | + | 0 | + | 28 | − | 31 | = | 25 |
| D | 0 | + | 28 | + | 24 | + | 0 | − | 31 | = | 21 |

# Shapley Allocation Method

**Theorem**

*The Shapley method is the only allocation method that is efficient, unbiased, subsidy free, and additive.*

| Order | Marginal Contribution | | | |
|---|---|---|---|---|
| | A | B | C | D |
| ABCD | 0 | 24 - 0 = 24 | 108 - 24 = 84 | 140 - 108 = 32 |
| ABDC | 0 | 24 - 0 = 24 | 140 - 96 = 44 | 96 - 24 = 72 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| BACD | 24 - 0 = 24 | 0 | 84 - 0 = 84 | 140 - 108 = 32 |
| BADC | 24 - 0 = 24 | 0 | 140 - 96 = 44 | 96 - 24 = 72 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| DCAB | 84 - 0 = 84 | 140 - 84 = 56 | 0 - 0 = 0 | 0 |
| DCBA | 140 - 0 = 140 | 0 - 0 = 0 | 0 - 0 = 0 | 0 |
| Average | 1464/24 = 61 | 792/24 = 33 | 600/24 = 25 | 504/24 = 21 |

# Learning Objectives

- Identify assumptions first
- Use algorithms and deductive arguments
- Use properties to compare methods

# Homework

- Read the analysis of **EPA** using the nucleolus method.
- Model the **New Product** scenario as a coalition game.
- Analyze the **New Product** coalition game.

# Conclusions

- Game theory is a great topic for building quantitative literacy.
- It is important to play the games and engage in modeling.
- Material is available to support this approach.
- Instructor engagement and creativity is both productive and fun.

# The Last Slide

- Questions?
- Instructor: David Housman, Goshen College
- Web-site: www.goshen.edu/dhousman and scroll to A Game Theory Path To Quantitative Literacy
- Email: dhousman@goshen.edu
- Book: Models of Conflict and Cooperation by Rick Gillman and David Housman, American Mathematical Society
- Assessment
- Thank you!

# Homework Four

- Read the online analysis of **EPA** using the nucleolus method.
- Model the **New Product** scenario as a coalition game.
- Analyze the **New Product** coalition game.

### New Product

Adonis, Beautex, and Celestron are three companies planning to collaborate on a new product. Adonis estimates that it could develop the product on its own and achieve a profit of \$2 million, but neither Beautex nor Celestron thinks that either could develop the product independently and turn a profit. Together the three companies estimate that they could develop the product and obtain a profit of \$50 million. Adonis and Beautex could obtain a profit of \$38 million without Celestron. Adonis and Celestron could obtain a profit of \$38 million without Beautex. Beautex and Celestron could obtain a profit of \$18 million without Adonis. How should the three companies allocate the profits from the new product?

# Nucleolus method analysis of EPA

Two serious objections can be made to the fairness of the Shapley method. First, the Shapley method is not rational: sometimes a coalition may receive less from the Shapley allocation than they could obtain on their own. Second, while the additive property seems reasonable, the games used in conjunction with the additive property seem somewhat artificial. Perhaps it would be more reasonable to compare the original game allocation with allocations obtained in games more naturally related to the original game. Our goal in this section is to describe another allocation method, called the nucleolus, that responds to our two objections.

Recall the **EPA** game, whose coalition gains are repeated here.

| EPA | |
|---|---|
| Coalition | Gain |
| ABCD | 140 |
| ABC | 108 |
| ABD | 96 |
| ACD | 84 |
| AB | 24 |
| other | 0 |

The Shapley allocation is $(61, 33, 25, 21)$.

While the Shapley method has an economic interpretation (marginal contributions of players), the nucleolus has a political interpretation (actual coalition gain). Suppose that the Shapley allocation $(61, 33, 25, 21)$ was proposed. Coalition ABC could obtain 108 on its own but actually receives $61+33+25 = 119$. Certainly, coalition ABC should be happy about this. One way to quantify coalition ABC's happiness is the additional amount received over what ABC could obtain on its own: $119-108 = 11$. In general, a coalition's happiness with a proposed allocation will be modeled with the difference between what the coalition receives and what the coalition could obtain on its own. We will call this difference the coalition's *excess* with the proposed allocation (because it is the gain the coalition receives in excess of what it could obtain on its own). The following table shows the calculation of coalition excesses with the Shapley allocation $(61, 33, 25, 21)$.

| Coalition | Allocation to | | | | | | | − | Gain | = | Excess |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | A | | B | | C | | D | | | | |
| ABC | 61 | + | 33 | + | 25 | | | − | 108 | = | 11 |
| ABD | 61 | + | 33 | | | + | 21 | − | 96 | = | 19 |
| ACD | 61 | | | + | 25 | + | 21 | − | 84 | = | 23 |
| BCD | | | 33 | + | 25 | + | 21 | − | 0 | = | 79 |
| AB | 61 | + | 33 | | | | | − | 24 | = | 70 |
| AC | 61 | | | + | 25 | | | − | 0 | = | 86 |
| AD | 61 | | | | | + | 21 | − | 0 | = | 82 |
| BC | | | 33 | + | 25 | | | − | 0 | = | 58 |
| BD | | | 33 | | | + | 21 | − | 0 | = | 54 |
| CD | | | | + | 25 | + | 21 | − | 0 | = | 46 |
| A | 61 | | | | | | | − | 0 | = | 61 |
| B | | | 33 | | | | | − | 0 | = | 33 |
| C | | | | + | 25 | | | − | 0 | = | 25 |
| D | | | | | | + | 21 | − | 0 | = | 21 |

Notice that the last four excesses duplicate the allocations to the individual players.

All coalitions have a positive excess, which means that all coalitions are happy. However, coalition ABC has the smallest excess, and so it is the coalition that is least happy with the Shapley allocation. Our goal will be to make the least happy coalition as happy as possible. This could be considered a fairness idea: help those who are least well off.

This also has a political interpretation: suppose that the game is being played by voters for a political office. Then the office holder, through his or her decisions and votes, allocates gains to the voters. Each coalition could be considered a special interest group, and the coalitions with the smallest excess are likely to make the most noise in the media against the office holder. Thus the office holder may want to make decisions and vote in a manner that will make the least happy coalition as happy as possible.

The following table shows the coalition excesses with six different proposed allocations (the allocations can be recovered by looking at the last four rows of excesses). Below the table, we explain how we came to choose these six allocations.

| Coalition | Gain | E1 | E2 | E3 | E4 | E5 | E6 |
|---|---|---|---|---|---|---|---|
| ABC | 108 | 11 | 14 | 17 | 16 | 16 | 16 |
| ABD | 96 | 19 | 18 | 17 | 18 | 22 | 22 |
| ACD | 84 | 23 | 22 | 21 | 21 | 21 | 28 |
| BCD | 0 | 79 | 78 | 77 | 77 | 73 | 66 |
| AB | 24 | 70 | 72 | 74 | 74 | 78 | 78 |
| AC | 0 | 86 | 88 | 90 | 89 | 89 | 96 |
| AD | 0 | 82 | 80 | 78 | 79 | 83 | 90 |
| BC | 0 | 58 | 60 | 62 | 61 | 57 | 50 |
| BD | 0 | 54 | 52 | 50 | 51 | 51 | 44 |
| CD | 0 | 46 | 44 | 42 | 42 | 38 | 38 |
| A | 0 | 61 | 62 | 63 | 63 | 67 | 74 |
| B | 0 | 33 | 34 | 35 | 35 | 35 | 28 |
| C | 0 | 25 | 26 | 27 | 26 | 22 | 22 |
| D | 0 | 21 | 18 | 15 | 16 | 16 | 16 |

In column E1, we arbitrarily started with the Shapley allocation $(61, 33, 25, 21)$. Coalition ABC has the smallest excess. To increase ABC's excess (to make ABC happier), we need to increase the payoffs to ABC. The only possible way to increase the payoffs to ABC is to reduce the payoff to D. As a first experiment, we will shift 3 from D to ABC.

To construct column E2, we shifted 3 from D equally to A, B, and C to obtain the second proposed allocation $(61, 33, 25, 21) + (1, 1, 1, -3) = (62, 34, 26, 18)$. Notice that coalition ABC still has the smallest excess, but the smallest excess has increased from 11 to 14; we are moving in the right direction. At the same time the excesses for coalitions ABD and D have both decreased to 18; we don't like making some coalitions less happy, but it is most important to make the least happy coalition as happy as possible.

In column E3, we again increase the excess of ABC by shifting 3 more from D to A, B, and C to obtain the third allocation $(62, 34, 26, 18) + (1, 1, 1, -3) = (63, 35, 27, 15)$. Coalition ABC's excess increased from 14 to 17, but D's excess decreased from 18 to 15. Coalition D now has the smallest excess, and so it is necessary to shift some payoff back to D. How much? Let's make the ABC and D excesses the same, which can be accomplished by shifting 1 to D.

In column E4, we shifted 1 to D from the arbitrarily chosen C. We obtain the allocation $(63, 35, 27, 15) + (0, 0, -1, 1) = (63, 35, 26, 16)$. The smallest excess is now 16. It is not possible to increase the smallest excess any further because to increase ABC's excess above 16 would require a transfer from D to the players in ABC, to increase D's excess above 16 would require a transfer to D from the players in ABC, and it is impossible to both transfer from and to D simultaneously. Although it is impossible to make the smallest excess greater than 16, notice that the third smallest excess with $(63, 35, 26, 16)$, held by coalition ABD, is 18 (the two 16s are considered the first and second smallest excesses). It would be possible to increase ABD's excess, without changing ABC's excess or D's excess, by shifting from C to A and/or B. Again, we don't want C's excess to become smaller than ABD's excess. So, we just shift half of the current difference in excesses $(26 - 18)/2 = 4$ from C.

In column E5, because of the low happiness of ACD, we will transfer 4 from C to A to obtain the fifth allocation $(63, 35, 26, 16) + (4, 0, -4, 0) = (67, 35, 22, 16)$. Notice that it is now impossible to increase both ABD's excess and C's excess any further because that would require that C receive both more and less than 22. Unfortunately, the third smallest excess for $(67, 35, 22, 16)$, as can be seen from E5, is now 21 associated with coalition ACD, not the 22 associated with coalitions ABD and C. Fortunately, ACD's excess can be reduced without changing ABC's, D's ABD's, and C's excesses, by shifting from B to A. Again, we don't want B's excess to become smaller than ACD's excess. So, we just shift half of the current difference in excesses $(35 - 21)/2 = 7$ from B to A.

Column E6 displays the final allocation $(67, 35, 22, 16) + (7, -7, 0, 0) = (74, 28, 22, 16)$. Because D cannot both give and receive, the smallest excesses, held by ABC and D, cannot be increased. Because C cannot both give and receive, the next smallest excesses, held by ABD and C, cannot be increased. Because B cannot both give and receive, the next smallest excesses, held by ACD and B, cannot be increased. We have now argued that to successively maximize the smallest coalition excesses, the payoffs to D, C, and B must be 16, 22, and 28, respectively. Since we want to allocate 140, player A must receive $140 - (16 + 22 + 28) = 74$.

The efficient allocation $(74, 28, 22, 16)$ successively maximizes the smallest excesses: the least happy coalition is as happy as possible; given that, the second least happy coalition is as happy as possible; given that, the third least happy coalition is as happy as possible; and so forth. This allocation method was first described by David Schmeidler in 1969.

**Nucleolus Method** The *nucleolus* is the efficient allocation that successively maximizes the smallest excesses.

# Homework One

- Play and analyze **Poison**

If the number of tiles is congruent to 0 or 2 mod 3, a winning strategy for Firstus is: Initially take two tiles if the number tiles is congruent to 0 mod 3, and take one tile if the number of tiles is congruent to 2 mod 3. On all subsequent moves, do the opposite of Secondus' last move.

If the number of tiles is congruent to 1 mod 3, Secondus' winning strategy is to always do the opposite of Firstus' last move.

- Play and analyze **Hex**

Although John Nash proved that Firstus always has a winning strategy for **Hex,** no general description of this strategy is known. Using computers, specific winning strategies have only been described for small boards.

- Create and analyze variations of **Nim**, **Poison**, and **Hex**

There are many different answers to this exercise, including changing the legal moves or the initial configurations of the game, or the number of players involved.

# Homework Two

- Model and analyze the **Matches** scenario.

Here is a possible cardinal payoff matrix for **Matches:**

| Matches Cardinal Payoffs | | Colin TENNIS | SOCCER |
|---|---|---|---|
| Rose | TENNIS | ($\boxed{10}$, $\boxed{6}$) | $(2, 5)$ |
| | SOCCER | $(0, 0)$ | ($\boxed{9}$, $\boxed{10}$) |

From the best response diagram shown, it is clear that (TENNIS, SOCCER) and (SOCCER, TENNIS) are two pure strategy Nash equilibria with corresponding payoffs $(10, 6)$ and $(6, 10)$, respectively. By choosing $(1-p)$TENNIS + $p$SOCCER, Rose has Colin choosing between TENNIS with a payoff of $6(1-p)+0p$ and SOCCER with a payoff of $5(1-p)+10p$. If Rose chooses $p$ to satisfy $6(1-p)+0p = 5(1-p)+10p$, which implies $p = \frac{1}{11}$, then any strategy for Colin is a best response. By choosing $(1-q)$TENNIS + $q$SOCCER, Colin has Rose choosing between TENNIS with a payoff of $10(1-q)+2q$ and SOCCER with a payoff of $0(1-q)+9q$. If Colin chooses $q$ to satisfy $10(1-q)+2q = 0(1-q)+9q$, which implies $q = \frac{10}{17}$, then any strategy for Rose is a best response. Thus, ($\frac{10}{11}$TENNIS + $\frac{1}{11}$SOCCER, $\frac{7}{17}$TENNIS + $\frac{10}{17}$SOCCER) is a Nash equilibrium with expected payoffs $(\frac{90}{17}, \frac{60}{11})$.

By choosing $(1-p)$TENNIS + $p$SOCCER, Rose will receive a payoff between $10(1-p)+0p = 10-10p$ if Colin chooses TENNIS and $2(1-p)+9p = 2+7p$ if Colin chooses SOCCER. The minimum Rose will receive is $2+7p$ for small values of $p$ and $10-10p$ for larger values of $p$. Rose's prudential strategy occurs when $p$ satisfies $2+7p = 10-10p$, which implies $p = \frac{8}{17}$. Hence, Rose's prudential strategy is $\frac{9}{17}$TENNIS + $\frac{8}{17}$SOCCER, and her security level is $\frac{90}{17}$. By choosing $(1-q)$TENNIS + $q$SOCCER, Colin will receive a payoff between $6(1-q)+5q = 6-q$ if Rose chooses TENNIS and $0(1-q)+10q = 10q$ if Rose chooses SOCCER. The minimum Colin will receive is $10q$ for small values of $q$ and $6-q$ for larger values of $q$. Colin's prudential strategy occurs when $q$ satisfies $10q = 6-q$, which implies $q = \frac{6}{11}$. Hence, Colin's prudential strategy is $\frac{5}{11}$TENNIS + $\frac{6}{11}$SOCCER, and his security level is $\frac{60}{11}$.

- Model and analyze the **Soccer Penalty Kicks** scenario.

Since there are only two possible outcomes (score or not score), the cardinal payoffs to the kicker can be set proportional to his probability of scoring.

| Soccer Penalty Kicks Cardinal Payoffs | | Goalie LEFT | RIGHT |
|---|---|---|---|
| Kicker | LEFT | $(1, -1)$ | $(10, -10)$ |
| | RIGHT | $(8, -8)$ | $(0, 0)$ |

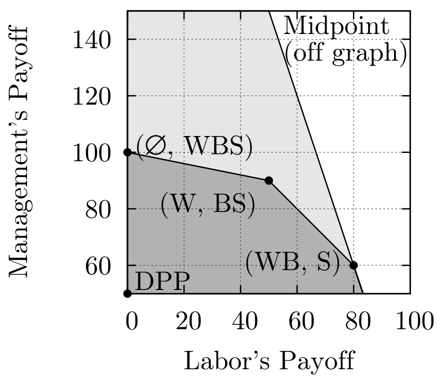
There is no pure strategy Nash equilibrium. By choosing $(1-p)$LEFT + $p$RIGHT, the kicker has the goalie choosing between LEFT with a payoff of $-1(1-p)-8p$ and RIGHT with a payoff of $-10(1-p)+0p$. If the kicker chooses $p$ to satisfy $-1(1-p)-8p=-10(1-p)+0p$, which implies $p=\frac{9}{17}$, then any strategy for the goalie is a best response. By choosing $(1-q)$LEFT + $q$RIGHT, the goalie has the kicker choosing between LEFT with a payoff of $1(1-q)+10q$ and RIGHT with a payoff of $8(1-q)+0q$. If the goalie chooses $q$ to satisfy $1(1-q)+10q=8(1-q)+0q$, which implies $q=\frac{7}{17}$, then any strategy for the kicker is a best response. Thus, ($\frac{8}{17}$LEFT + $\frac{9}{17}$RIGHT, $\frac{10}{17}$LEFT + $\frac{7}{17}$RIGHT) is a Nash equilibrium with expected payoffs $(\frac{80}{17},-\frac{80}{17})$.

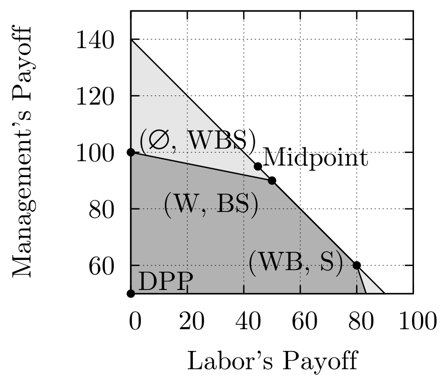
By choosing $\frac{8}{17}$LEFT + $\frac{9}{17}$RIGHT, the kicker obtains $\frac{80}{17}$ no matter what the goalie does. If the goalie chooses $\frac{10}{17}$LEFT + $\frac{7}{17}$RIGHT, then the kicker cannot obtain more than $\frac{80}{17}$. Hence, $\frac{8}{17}$LEFT + $\frac{9}{17}$RIGHT must be prudential for the kicker with a security level of $\frac{80}{17}$. By choosing $\frac{10}{17}$LEFT + $\frac{7}{17}$RIGHT, the goalie obtains $-\frac{80}{17}$ no matter what the kicker does. If the kicker chooses $\frac{8}{17}$LEFT + $\frac{9}{17}$RIGHT, then the goalie cannot obtain more than $-\frac{80}{17}$. Hence, $\frac{10}{17}$LEFT + $\frac{7}{17}$RIGHT must be prudential for the kicker with a security level of $-\frac{80}{17}$.

# Homework Three

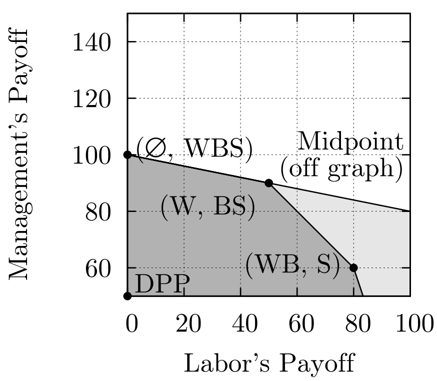
- Analyze **Acme Industrial** using the Nash method, described online.

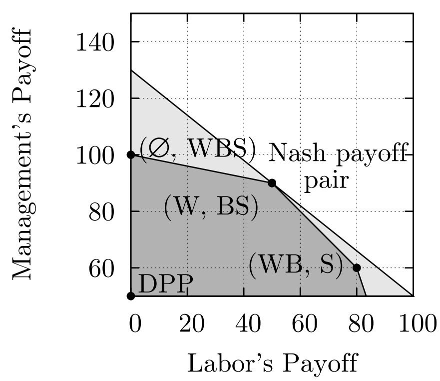
We now illustrate the Nash method with **Acme Industrial**. For our first step, we use the right-most efficient and rational line (ACME LOWER) $y = -3x + 300$. From the graph, it is clear that the hypotenuse midpoint is to the left of the feasible payoff pairs on the line. So for our second step, we use (ACME MIDDLE) $y = -x + 140$, which has intersections with the disagreement payoff lines $x = 0$ and $y = 50$ at $(0, 140)$ and $(90, 50)$, making the midpoint of the hypotenuse $\frac{1}{2}(0, 140) + \frac{1}{2}(90, 50) = (45, 95)$. We see again that this point is to the left of the rational payoff pairs on the line.




So for our third step, we use (ACME UPPER) $y = -0.2x + 100$. From the graph, it is clear that the midpoint of the hypotenuse is to the right of the rational payoff pairs on the line. With this switch in sides, the Nash payoff pair is identified as the payoff pair shared by the lines in steps 3 and 2: $(50, 90)$. This payoff pair can be implemented by having Labor win on the wage issue and having Management win on the benefits and security issues.




Although unnecessary for the completion of the Nash method, the step 4 graph includes the right triangle that justifies our selection of $(50, 90)$ for the Nash payoff pair. That is, the disagreement payoff lines intersection points $(0, 130)$ and $(100, 50)$ were chosen so that $(50, 90)$ would be their midpoint.

- Model the **Mating** scenario as a bargaining game.

One option is to use $(0,0)$ as the Disagreement Payoff Pair (DPP). Then the ordered pairs in the payoff matrix

| Mating Cardinal Payoffs | | Male FAITHFUL | PHILANDERING |
|---|---|---|---|
| Female | COY | $(2,2)$ | $(0,0)$ |
| | FAST | $(5,5)$ | $(-5,15)$ |

can be used to describe a region of feasible solutions.

- Analyze the **Mating** bargaining game.

Using $(0,0)$ as the DPP, the egalitarian solution is $(5,5)$, the Raiffa solution is $(5/2, 15/2)$, and the Nash solution is also $(5,5)$.

# Homework Four

- Read the online analysis of **EPA** using the nucleolus method.

See separate document.

- Model the **New Product** scenario as a coalition game.

The players are the three companies, which will be represented by the first letters of their names. The gain of a coalition is synonymous with the profit generated by that coalition.

| New Product | |
|---|---|
| Coalition | Gain |
| ABC | 50 |
| AB | 38 |
| AC | 38 |
| BC | 18 |
| A | 2 |
| B | 0 |
| C | 0 |

- Analyze the **New Product** coalition game.

If the companies decide to allocate to use the Shapley method, then the allocation can be computed from the following table.

| Order | Marginal Contribution | | |
|---|---|---|---|
| | A | B | C |
| ABC | 2 | 36 | 12 |
| ACB | 2 | 12 | 36 |
| BAC | 38 | 0 | 12 |
| BCA | 32 | 0 | 18 |
| CAB | 38 | 12 | 0 |
| CBA | 32 | 18 | 0 |
| Average | 24 | 13 | 13 |

We use our heuristic of informed guessing to find the nucleolus for **New Product**, whose gains are repeated in the table below.

| New Product | |
|---|---|
| Coalition | Gain |
| ABC | 50 |
| AB | 38 |
| AC | 38 |
| BC | 18 |
| A | 2 |
| B | 0 |
| C | 0 |

As can be seen in the table below, we first examine the Shapley allocation $(24, 13, 13)$, and find that coalitions AB and AC have the smallest excesses. A negative excess indicates that the corresponding coalition receives less than it could obtain on its own. In order to increase both excesses, the payoff to A must be increased at an expense to B and C. Since B and C are indistinguishable except by their name, we keep the payoffs to B and C the same.

| Coalition | Gain | Excess With | | |
|---|---|---|---|---|
| | | $(24, 13, 13)$ | $(26, 12, 12)$ | $(30, 10, 10)$ |
| AB | 38 | $-1$ | 0 | 2 |
| AC | 38 | $-1$ | 0 | 2 |
| BC | 18 | 8 | 6 | 2 |
| A | 2 | 22 | 26 | 30 |
| B | 0 | 13 | 12 | 10 |
| C | 0 | 13 | 12 | 10 |

The first change is to take 1 from each of B and C and give 2 to A to obtain $(24, 13, 13) + (2, -1, -1) = (26, 12, 12)$. This increases the first and second smallest excesses from $-1$ to 0 and lowers the third smallest excess from 8 to 6. Extrapolating these changes, we can see that by shifting 2 more from each of B and C to A, the three smallest excesses should become equal with $(26, 12, 12) + (4, -2, -2) = (30, 10, 10)$.

It is not so clear with this example whether there is a way to increase the smallest excess further. However, this can be handled algebraically. Suppose $(a, b, c)$ is an efficient allocation whose minimum excess is at least 2. In particular, the AB, AC, and BC excesses must be each at least 2, which can be written

$$\begin{aligned} a + b - 38 &\geq 2 \\ a + c - 38 &\geq 2 \\ b + c - 18 &\geq 2 \end{aligned}$$

Summing these three inequalities and using efficiency, we obtain

$$\begin{aligned} 2a + 2b + 2c - 94 &\geq 6 \\ 2(a + b + c) &\geq 100 \\ 2(50) &\geq 100 \\ 100 &\geq 100 \end{aligned}$$

Since 100 is not strictly greater than 100, it must be the case that all previous inequalities must hold with equality. That is,

$$\begin{aligned} a+b-38 &= 2 \\ a+c-38 &= 2 \\ b+c-18 &= 2 \end{aligned}$$

This system of three equations in the three unknowns $a$, $b$, and $c$, has a unique solution, $(a, b, c) = (30, 10, 10)$. This verifies that $(30, 10, 10)$ is the nucleolus.

**A Game Theory Path to Quantitative Literacy**
**Math Fest 2008 Minicourse**
**Rick Gillman and David Housman**
**Bibliography**

There are many game theory books available with a variety of intended audiences. Many are directed towards graduate or upper-level undergraduate economics majors, whose mathematical sophistication would be higher than the students in a quantitative literacy course. Unlike calculus, there is not a standard curriculum for game theory courses. So, when deciding upon a book to use, you should consider both the mathematical level and the topics covered.

The following are topics often covered in a first game theory course at the undergraduate level (roughly equivalent names are given in parentheses). No course or book would cover all of these topics. The focus of the minicourse was on the four *ed topics.

- Player preferences: ordinal, cardinal, and ratio scales.
- *Deterministic games (combinatorial games, zero-sum extensive form games with perfect information): players take turns with perfect information and no randomness, and the only possible outcomes are win, lose, and tie. These games can be solved by backward induction. With two-players, we have Zermelo's Theorem. Examples include Nim, Hex, Chess, and Trickster.
- *Strategic games (normal form games, matrix games): players simultaneously choose strategies which jointly determine the outcome. Solutions include Nash equilibrium, prudential strategies, dominance, efficient, and evolutionarily stable. There is a jump in sophistication when mixed strategies and/or more than two players are considered. There are straight-forward algebraic methods for finding mixed strategy solutions when there are only two players and one player has only two strategies. Examples include Mating, Matches, and Prisoners' Dilemma.
- Two-player zero-sum games: special case of strategic games in which there are exactly two players and the sum of the payoffs is zero. The Nash equilibria and prudential strategy pairs coincide and are often called minimax or maximin strategies. These solutions can be found using a linear program, which may be solved using the simplex algorithm. Examples include Rock, Paper, Scissors; Morra; and Matching Pennies.
- Repeated games: a strategic game that is repeated with the overall payoff to a player a discounted sum of the single period games. The primary example is Prisoners' Dilemma where it can be shown that a TIT FOR TAT strategy can be a Nash equilibrium, resulting in player cooperation in each period.
- Sequential games (extensive form games with perfect information): players take turns with perfect information and no randomness. These games can be solved by backward induction. Examples include Cuban Missile Crisis.
- Imperfect information games (extensive form games, Bayesian games, stochastic games): players can have private information and can make secret moves. The equilibrium concepts become more complicated and dependent upon a good understanding of conditional probability. Examples include auctions.
- Experiments: students are encouraged to play the games and some of the literature about how people actually play games is explored.
- *Bargaining games (bargaining problem): players can make binding agreements to choose specific strategies in order to obtain agreed upon payoffs. Three solution concepts are egalitarian, Raiffa/Kalai/Smorodinsky, and Nash. These solutions can be deductively characterized by fairness properties. Examples include labor-management negotiations and any strategic game if players are offered the opportunity to communicate and enter into contracts.
- *Coalition games (characteristic form games): three or more players can make binding agreements to split amongst themselves a payoff, which varies by the coalition of players that forms. Solutions

concepts include the Shapley value, the nucleolus, the core (group rational), and stable sets. Solutions can be deductively characterized by fairness properties. Examples include sharing the savings from a joint venture.

- Voting games (simple games): coalition games in which winning coalitions can split one and losing coalitions receive zero. Solution concepts now measure player voting power, and they include the Shapley-Shubik, Banzhaf, Johnson, and Deegan-Packel power indices.
- Fair division games (fair division problems, cake-cutting): players, who have equal shares in one or more objects, allocate the objects. The objects may be divisible (e.g., a cake) or indivisible (e.g., a sculpture), and players may be willing to transfer money amongst themselves. Solutions can be deductively characterized by fairness properties, and strategic aspects of solutions are often examined. Examples include dividing an inheritance or a cake.
- Mathematics vs. modeling and applications. Some books focus more on the mathematics (e.g., use the simplex algorithm to find minimax solutions in two-player zero-sum games, prove the existence of a Nash equilibrium in every finite strategic game, or prove the Nash bargaining solution is the unique efficient, unbiased, scale invariant, and independent of irrelevant payoff pairs solution). Some books focus more on how to apply the game theory models and solutions to specific applications. Some books give students guidance for how to model a new scenario as a game.

We describe below books that could be used in a game theory course intended for undergraduate students. There are only a few we would recommend for a quantitative literacy or liberal arts mathematics course. There are many more books available for the more mathematically sophisticated student. While such books would be inappropriate for students with a weak mathematics background, they may be good books for an instructor to browse while preparing to teach a game theory course. We have not included books primarily intended for graduate students.

## For Quantitative Literacy and Liberal Arts Courses

Avinash Dixit and Susan Skeath, *Games of Strategy*, W. W. Norton and Company focuses on strategic and sequential games with a variety of applications. It also covers imperfect information games, Prisoners' Dilemma repeated games, evolutionary strategies, voting games, and bargaining games.

Rick Gillman and David Housman, *Models of Conflict and Cooperation*, unpublished manuscript written expressly to build quantitative literacy skills. This book emphasizes the modeling process including the meaning of ordinal and cardinal utility. The three chapters on cooperative games emphasize the characterization of solutions by fairness properties. It covers deterministic games, player preferences, strategic games including mixed strategies, experimental games, repeated Prisoners' Dilemma, bargaining games, coalition games, and fair division games with indivisible objects and money.

Philip D. Straffin, *Game Theory and Strategy*, Mathematical Association of America follows a traditional order with some very nice applications. It covers two-player zero-sum games, strategic games including strategic moves and evolutionarily stable strategies, sequential games, repeated Prisoners' Dilemma, voting games, coalition games, and bargaining games.

Alan D. Taylor, *Mathematics and Politics: Strategy, Voting, Power and Proof*, Springer-Verlag has a somewhat less standard selection of topics including deductively examining social choice functions by fairness properties (nicely done) and a characterization of when a simple game is weighted. It also covers strategic games, sequential games, and voting games.

## For More Advanced Undergraduate Students

Edward Bolger, *Topics in Game Theory*, Outskirts Press is a good traditional text that covers two-player zero-sum games including solving by the simplex algorithm, coalition games, voting games, and multichoice games. The exposition is terse given that these are intended to be lecture notes more than a text.

Prajit K. Dutta, *Strategies and Games: Theory and Practice*, Massachusetts Institute of Technology focuses on economics applications, describes the most realistic examples, and provides the most in-depth coverage of strategic games. It covers player preferences, strategic games, sequential games, zero-sum games, imperfect information games, and repeated games.

Herbert Gintis, *Game Theory Evolving: A Problem-Centered Introduction to Modeling Strategic Interaction*, Princeton University Press has a great breadth of applications and, unlike in most books where the exercises occupy a small proportion of the text and come at the end of each section or chapter, this book focuses on problems. It covers strategic games especially evolutionarily stable strategies, sequential games, and bargaining games.

Nolan McCarty and Adam Meirowitz, *Political Game Theory: An Introduction*, Cambridge University Press covers player preferences, social choice theory, strategic games, sequential games, imperfect information games, repeated games, and bargaining games.

Elliot Mendelson, *Introducing Game Theory and Its Applications*, Chapman & Hall/CRC is a good traditional text with coverage of linear programming. It covers deterministic games, two-player zero-sum games, simplex algorithm, strategic games, Nash bargaining solution, and coalition games (Shapley value and stable set).

Michael Mesterton-Gibbons, An *Introduction to Game-Theoretic Modeling*, Addison-Wesley has the applications drive the mathematics. You won't see much in the way of proofs or zero-sum games. However, you will see more realistic examples, evolutionary stability, bargaining games, coalition games and power indices, prisoners' dilemma and repeated play.

James D. Morrow, *Game Theory for Political Scientists*, Princeton University Press focuses on player preferences, sequential games, imperfect information games, and repeated games. It also briefly covers strategic games.

Martin Osborne, *An Introduction to Game Theory*, Oxford University Press is a broad introduction with many applications for an upper-level economics student. It covers strategic games, sequential games, coalition games (core), imperfect information games, zero-sum games, evolutionarily stable strategies, repeated games, and bargaining games.

Saul Stahl, *A Gentle Introduction to Game Theory*, American Mathematical Society focuses on the mathematics of finding minimax solutions for two-player zero-sum games. It also briefly covers finding nonzero-sum game Nash equilibrium.