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Fair Allocation Methods for Coalition Games

David Housman

ABSTRACT. A coalition game is a mathematical model of situations in which players can make enforceable agreements to cooperate. For each set of players, there is a numerical gain that can be distributed should the players agree to cooperate. This paper describes some interesting classes of coalition games, methods for allocating the gains among the players, and properties that formalize intuitive notions of fairness. We determine on which class of games each method satisfies each property. Some methods are characterized via properties.

Game theory uses mathematical models to explore situations in which two or more decision makers, the players, have an effect on outcomes that each player may value differently. A coalition game is an austere model of situations in which players can make enforceable agreements to cooperate. If some players agree to cooperate, they must know, at minimum, what can be accomplished by their cooperation. A coalition game uses numbers to describe what all sets of players can accomplish through cooperation. These numbers can be thought of as money, utility, value, or gain that can be distributed among the cooperating players.

Given a coalition game, we can ask what *will* happen? Presumably, players will negotiate a distribution of the obtainable gains that is acceptable to each of the cooperating players. We can also ask what *should* happen? For example, an external arbitrator may be asked to impose a distribution of the obtainable gains. Fairness will be part of the answer to either question. An arbitrator is supposed to impose a fair distribution, and in a negotiation, a player is unlikely to agree to a distribution that does not seem fair.

This article will describe coalition games, methods to distribute obtainable gains, and method properties that formalize intuitive notions of fairness. We will see which methods satisfy which properties and then characterize some methods with properties. One goal is to provide an introduction to, not a complete survey of, the literature. Some of the results are new, but most of the results are in the literature. A second goal is to suggest directions for future research that could be accomplished by undergraduate or graduate students of mathematics. Good

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sources to obtain further pointers into the appropriate literature are [**1**, chapters 13, 17, and 18], [**2**, chapters 34 and 36], [**3**, chapters 53-56], [**8**], [**21**], [**22**, chapter 7], [**23**, chapter 5], [**25**], [**32**], and [**37**].

Section 1 defines coalition game and allocation, describes a few examples and classes of coalition games, and describes a few fairness properties for allocations. Section 2 describes several fairness properties for allocation methods and illustrates these properties with a very simple allocation method. Sections 3, 4, and 5 describe many reasonable allocation methods and determine which properties each method satisfies. Section 6 states a mutual incompatibility of three properties and three characterizations of allocation methods via properties. Section 7 suggests directions for future research.

1. Coalition Games

This section defines coalition game and allocation, describes a few examples and classes of coalition games (the containment relationships among these are summarized in Figure 1), and describes a few fairness properties for allocations.

Definition 1.1 (Coalition Game). A *coalition game* consists of a finite set N and a real-valued function w from the subsets of N that satisfies $w(\varnothing) = 0$. An element i in N is called a *player*, and a nonempty subset S of N is called a *coalition*.

The real number $w(S)$ is interpreted as the worth, value, utility, or gain of the coalition S, that is, the amount available to distribute among the players in S if that coalition forms.

Example 1.2 (Savings). The government has mandated improvements in the sewage treatment facilities in the cities of Avon, Barport, Claron, and Delmont. Each city could work separately, but \$140 million would be saved by all four working together. If one of the cities was unwilling to cooperate, some triples of cities could also save money: without Delmont's cooperation, Avon, Barport, and Claron could save \$108 million; without Claron's cooperation, Avon, Barport, and Delmont could save \$96 million; without Belmont's cooperation, Avon, Claron, and Delmont could save \$84 million; and without Claron's or Delmont's cooperation, Avon and Barport could save \$24 million. No other subset of the cities could save money over completing the projects individually. In particular, Barport, Claron, and Delmont cannot save any money without the assistance of Avon. This situation can be modeled with a coalition game involving the players $N = \{A, B, C, D\}$, where we represent each city with the first letter of its name, and the function w defined by $w(N) = 140$, $w(\{A, B, C\}) = 108$, $w(\{A, B, D\}) = 96$, $w(\{A, C, D\}) = 84$, $w(\{A, B\}) = 24$, and $w(S) = 0$ for all other coalitions S. In a standard abuse of notation, we usually remove the set brackets and commas in examples. For example, instead of $w(\{A, C, D\}) = 84$, we write $w(ACD) = 84$.

Remark 1.3. The set of games w with a fixed player set N can be viewed as vector in a $2^{|N|} - 1$ dimensional convex subset of $\mathbb{R}^{2^N}$, where 2^N is the set of subsets of N. So, it makes mathematical sense to add games using vector addition and multiply games by scalars.

Given a coalition game, our goal is to determine a fair distribution of the possible gains. Our presumption will be that all players either choose or are forced (say by government decree) to cooperate.

DEFINITION 1.4 (Allocation). Given a coalition game (N, w), an *allocation* is a real-valued vector x indexed by the players that satisfies the *efficiency condition* $\sum_{i \in N} x_i = w(N)$. The number x_i is called player i's *payoff*.

For the savings coalition game, $(20, 30, 40, 50)$ and $(100, 25, 25, -10)$ are allocations, although neither seems the least bit fair.

EXERCISE 1. *This is a good place for the reader to stop for a few minutes and do one or both of the following. First, find three other people and play the savings coalition game, that is, have each person be a representative for one of the four cities and then negotiate to an acceptable allocation. Second, act as an arbitrator and choose what you think is the most fair allocation.*

In this paper, we will typically consider games in which there is no disincentive for players to cooperate with other players.

DEFINITION 1.5 (Superadditive Game). A coalition game (N, w) is *superadditive* if $w(S) + w(T) \leq w(S \cup T)$ for all coalitions S and T satisfying $S \cap T = \varnothing$.

The savings game is superadditive. Indeed, if $S \cap T = \varnothing$, then at most one of S and T contains player A, and so $w(S) = 0$ or $w(T) = 0$. Since worths are non-decreasing in the cardinality of the coalition, $w(S) + w(T) = \max\{w(S), w(T)\} \leq w(S \cup T)$.

Player D would object to the fairness of $(100, 25, 25, -10)$ because player D could obtain a payoff of 0 without cooperating with anyone else. We posit that players will agree to an allocation only if each player receives at least as much as that player could receive on its own.

DEFINITION 1.6 (Player-Rational Allocation). Suppose (N, w) is a coalition game. The allocation x is *player-rational* if $x_i \geq w(\{i\})$ for all $i \in N$.

The savings game allocation $(20, 30, 40, 50)$ is player-rational, but players A, B, and C may object because they only receive $20+30+40 = 90$ but as a coalition they could obtain $w(ABC) = 108$. A coalition-rational allocation avoids such objections.

DEFINITION 1.7 (Coalition-Rational Allocation). Suppose (N, w) is a coalition game. The allocation x is *coalition-rational* if $\sum_{i \in S} x_i \geq w(S)$ for all $S \subseteq N$.

Clearly, a coalition-rational allocation is player-rational. The savings game allocations $(140, 0, 0, 0)$ and $(8, 56, 44, 32)$ are coalition-rational, but few would argue either is fair. Hence, more intuitions about fairness need to be formalized. At the same time, insisting on coalition-rational allocations may be too restrictive because some games have no coalition-rational allocation.

EXAMPLE 1.8 (Simple Majority). There are at least three players and

$$w(S) = \begin{cases} 1, & \text{if } |S| > |N|/2 \\ 0, & \text{otherwise} \end{cases}$$

This game has no coalition-rational allocations. Indeed, if x were a coalition-rational allocation, then $\sum_{i \in N \setminus \{j\}} x_i \geq 1$ for all $j \in N$. Summing these inequalities, we obtain $(n-1) \sum_{i \in N} x_i \geq n$, contradicting the efficiency condition $\sum_{i \in N} x_i = 1$.

DEFINITION 1.9 (Balanced Game). A coalition game (N, w) is *balanced* if there are coalition-rational allocations.

Since no coalition-rational allocation exists for a simple majority game, we cannot expect a fair allocation to be coalition-rational. In a negotiation, perhaps a coalition of just over half of the players will form and evenly split the gain while the other players are left with payoffs of zero. In an arbitration, there is no fair way to distinguish among the players, and so the same amount should be given to each player. We now formalize this intuition.

DEFINITION 1.10 (Unbiased Allocation). Suppose (N, w) is a coalition game. Players i and j are *indistinguishable* if $w(S \cup \{i\}) = w(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. The allocation x is *unbiased* if $x_i = x_j$ for all indistinguishable players i and j.

The unique unbiased allocation for a simple majority game gives a payoff of $1/|N|$ to each player. However, the unbiased property does not directly eliminate any allocations to the savings game. So, instead of only considering players that are indistinguishable, we strengthen the property by considering players that are clearly distinguishable. too.

DEFINITION 1.11 (Strongly Unbiased Allocation). Suppose (N, w) is a coalition game. Player i is called *weaker than* player j if $w(S \cup \{i\}) \leq w(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. The allocation x is *strongly unbiased* if $x_i \leq x_j$ whenever player i is weaker than player j.

In the savings game, player D is weaker than player C is weaker than player B is weaker than player A. So, the allocation $(50, 35, 35, 20)$ is strongly unbiased, but the allocation $(8, 56, 44, 32)$ is not strongly unbiased.

The superadditivity criterion can be strengthened to obtain games with very strong incentives for cooperation.

DEFINITION 1.12 (Convex Game). A coalition game (N, w) is *convex* if $w(S) + w(T) \leq w(S \cup T) + w(S \cap T)$ for all $S, T \subseteq N$.

The simple majority game is a game that has no coalition-rational allocation. On the other hand, convex games have "large" sets of coalition-rational allocations [**30**]. Games that are not convex may have coalition-rational allocations. For example, the savings game has the coalition-rational allocation $(140, 0, 0, 0)$, but it is not convex because $w(ABC) + w(ABD) \not\leq w(ABCD) + w(AB)$.

DEFINITION 1.13 (Simple Game). A coalition game (N, w) is *simple* if $w(N) = 1$ and $w(S) \in \{0, 1\}$ for all $S \subseteq N$. If $w(S) = 1$, then S is called *winning*, and if $w(S) = 0$, then S is called *losing*.

Simple games model voting systems, and a fair allocation is interpreted as the voting powers of the voters. The simple majority games are examples of simple games. Here is a more complex example of a simple game.

EXAMPLE 1.14 (Federal Law). The players are the 435 Representatives, 100 Senators, and President of the United States of America. We think of the Vice President, who can break ties among Senators, as a proxy for the President. In order for a proposal to become federal law, it must be approved by the President and simple majorities of the Representatives and Senators, or without approval of the President, it must be approved by two-thirds majorities of the Representatives and Senators. Hence, a coalition is winning if and only if it contains (1) at least 218 Representatives, at least 50 Senators, and the President; or (2) at least 290 of the

Representatives and at least 67 of the Senators. Intuitively, the President has more voting power than a Senator who has more voting power than a Representative.

A superadditive simple game corresponds to voting situations which satisfy the following natural conditions: (1) supersets of winning coalitions are winning, and (2) the complement of a winning coalition must be losing. Convex simple games are the unanimity games.

DEFINITION 1.15 (Unanimity Game). Suppose $T \subseteq N$. The *unanimity game* on T is the game (N, u^T) where

$$u^T(S) = \begin{cases} 1, \text{ if } T \subseteq S \\ 0, \text{ otherwise} \end{cases}$$

In words, a coalition is winning if and only if the coalition contains T. If $T = \{i\}$, then we call player i a *dictator*.

In the unanimity game (N, u^T), two players in T are indistinguishable, and two players in $N \backslash T$ are indistinguishable. Hence, an unbiased allocation gives the same payoff to each player in T and gives the same payoff to each player in $N \backslash T$. Since the players in $N \backslash T$ neither contribute nor detract from gains obtained by any coalition, it seems only fair that players in $N \backslash T$ receive nothing and pay nothing.

DEFINITION 1.16 (Subsidy-Free Allocation). Suppose (N, w) is a coalition game. Player i is called a *dummy* if $w(S) = w(S \backslash \{i\}) + w(\{i\})$ for all coalitions S containing i. Note that we are using the convention $w(\varnothing) = 0$. The allocation x is *subsidy-free* if $x_i = 0$ for all dummy players i.

The unique unbiased and subsidy-free allocation for the unanimity game (N, u^T) is x satisfying $x_i = 1/|T|$ if $i \in T$ and $x_i = 0$ otherwise. The same allocation is the unique unbiased and coalition-rational allocation.

Besides having uniquely determined fair allocations, the unanimity games on n-players also form a basis for the space of games on n-players. Indeed, the number of unanimity games, $2^n - 1$, is the same as the dimension of the space of games, and the space of games is spanned by the unanimity games: $w = \sum_{T \subseteq N} d_T u^T$ where $d_T = \sum_{S \subseteq T} (-1)^{|T|-|S|} w(S)$ **[29]**. The game w is convex if $d_T \geq 0$ for all coalitions T satisfying $|T| \geq 2$

The subsidy-free property is also useful in selecting the most reasonable allocation in another class of games.

DEFINITION 1.17 (Additive Game). If there exists a real-valued vector $z \in \mathbb{R}^N$ for which $w(S) = \sum_{i \in S} z_i$ for all $S \subseteq N$, then the coalition game (N, w) is *additive*.

The unique unbiased and subsidy-free allocation for the additive game (N, w) is x satisfying $x_i = w(\{i\})$. The same allocation is the unique player-rational allocation.

It is sometimes useful to compare allocations in different, but related, games.

EXAMPLE 1.18 (Veto Power). Suppose $N = \{1, 2, \ldots, n\}$, $k \in N$, and $2 \leq r \leq n - 1$. Define

$$v^{k,r}(S) = \begin{cases} 1, \text{ if } k \in S \text{ and } |S| \geq r \\ 0, \text{ otherwise} \end{cases}$$

If x is coalition-rational, then for each player $i \neq k$, $0 \leq x_i = 1 - \sum_{j \in N \backslash \{i\}} x_i \leq 0$, which implies $x_i = 0$, and so $x_k = 1$. Thus, the kth unit vector χ^k is the unique coalition-rational allocation.

The allocation $\chi^1 = (1, 0, \dots, 0)$ is the unique coalition-rational allocation in both the dictator game $u^{\{1\}}$ and the veto power games $v^{1,r}$. In the dictator game $u^{\{1\}}$, player 1 can obtain 1 without cooperating with the other players. In the veto power games $v^{1,r}$, player 1 needs to cooperate with at least $r-1$ other players to obtain 1. Restricting ourselves to coalition-rational allocations will not reflect the apparent differences in power player 1 has in these games.

EXAMPLE 1.19 (Cost Overrun). The savings game is based on cost estimates before the improvements have been started. Suppose that the four cities decide to cooperate and a \$20 million cost overrun occurs. Although a cost overrun might have occurred had a different coalition decided to cooperate, for simplicity, we will assume that is not the case. This situation can be modeled with the coalition game (N, w) where $N = \{A, B, C, D\}$, $w(N) = 120$, $w(ABC) = 108$, $w(ABD) = 96$, $w(ACD) = 84$, $w(AB) = 24$, and $w(S) = 0$ otherwise.

It seems reasonable that whatever the fair allocation is for the savings game, each player should receive less in the cost overrun game.

DEFINITION 1.20 (Zero-Normalized Game). The coalition game (N, w) is *zero-normalized* if $w(\{i\}) = 0$ for all $i \in N$.

Except for the unanimity games $u^{\{i\}}$, all of our examples have been zero-normalized. Another non-zero-normalized game could have been defined had we described the savings game differently. Presumably for each coalition S, there is a cost $c(S)$ for the cities in S to jointly improve their sewage treatment facilities. The savings game would have been obtained by computing $w(S) = \sum_{i \in S} c(\{i\}) - c(S)$. It would have been reasonable to instead define the game $v(S) = -c(S)$. This later game would not be zero-normalized. Nonetheless, it would seem that the allocation to player i in w should be $c(\{i\})$ plus the allocation to player i in v.

As suggested by our comparison of the savings and cost overrun games and our comparison of two ways of representing joint costs as a coalition game, a desire for consistent treatment of players across different games motivates the definition of allocation *methods*, the topic of the next section.

In this section, we have described two economics examples (savings and cost overrun), three political science examples (federal law, simple majority, and veto power), and seven classes of games (zero-normalized, superadditive, balanced, convex, simple, unanimity, and additive). The containment relationship among these is illustrated in Figure 1.

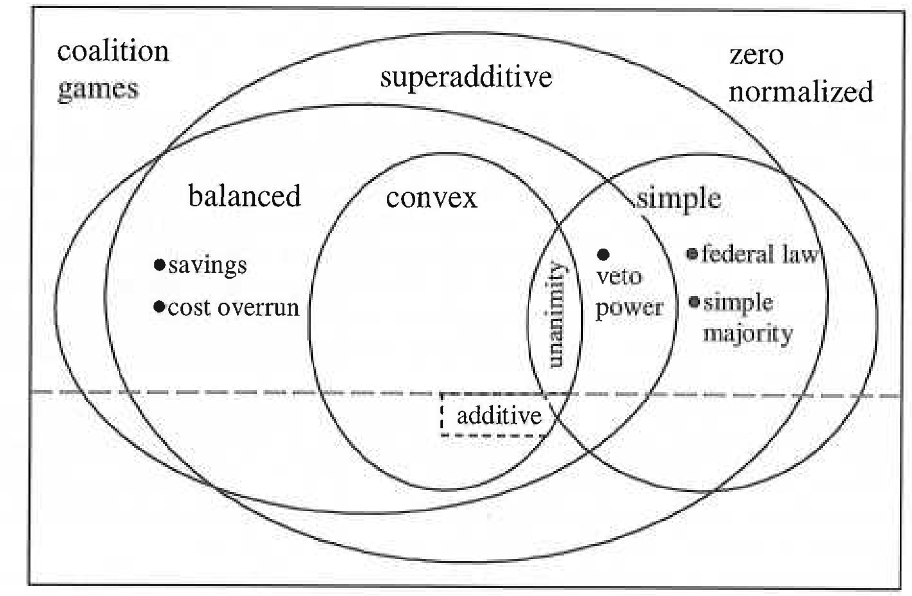

Figure 1. Containment relationships among the classes and examples of coalition games.

There are many other situations that can be modeled as coalition games. Lucas **[18]** describes several voting bodies including the United Nations Security Council, the Israeli Knesset, the United States Electoral College, and New York State county boards. Lucas and Billera **[19]** describe several economic games including airport landing fees, WATS telephone lines, waterway pollution abatement costs, and the sharing of the cost of a communication satellite. Moulin **[22]** describes coalition game models for bilateral assignment markets, marriage markets, output-sharing of production, exchange economies, and fair division with money. Young **[36]** describes a cost allocation problem among six municipalities in Sweden and airport landing fees. Moulin **[23]** models mail distribution and access to a network as coalition games. Driessen **[8]** models water resource development in Japan and bankruptcy as coalition games. A primary focus of Taylor and Zwicker **[32]** is the relationship between weighted voting games and simple games.

2. Allocation Methods and Properties

An allocation can seem fair if it is selected by a method whose description seems fair.

DEFINITION 2.1 (Allocation Method). An *allocation method* or *value* is a function from coalition games to allocations.

Here is a simple example of an allocation method.

DEFINITION 2.2 (Egalitarian Method). The *egalitarian method* gives player i her individual gain, $w(\{i\})$, and an equal share of any possible additional gain to be had by all players cooperating, $w(N) - \sum_{j \in N} w(\{j\})$. An explicit formula is

$$\xi_i(N,w) = w(\{i\}) + \frac{1}{|N|}\left(w(N) - \sum_{j \in N} w(\{j\})\right)$$

The egalitarian method selects $(35, 35, 35, 35)$ for the savings game and $(30, 30, 30, 30)$ for the cost overrun game; neither allocation is coalition-rational (coalition ABC receives less than 108 in both allocations). The game (N, w) where $N = \{1, 2, 3, 4\}$ and $w = 3u^{12} + u^{13}$ is convex, but $\xi(N, w) = (1, 1, 1, 1)$ is neither coalition-rational ($\xi_1(N, w) + \xi_2(N, w) < 3 = w(12)$) nor subsidy-free (player 4 is a dummy but does not receive $w(4) = 0$). For simple games without a dictator (which include the simple majority, veto power, and federal law games), the egalitarian method selects $(1/|N|, \ldots, 1/|N|)$, which is neither subsidy-free nor coalition-rational if $w(S) = 1$ for some $S \neq N$.

Although it often does not select coalition-rational nor subsidy-free allocations, the egalitarian method selects player-rational allocations at superadditive games, and always selects strongly unbiased allocations. Indeed, $w(N) \geq \sum_{j \in N} w(\{j\})$ for superadditive games implies that $\xi_i(N, w) \geq w(\{i\})$, and if player i is weaker than player j, $w(\{i\}) \leq w(\{j\})$ implies that $\xi_i(N, w) \leq \xi_j(N, w)$.

It is not enough to define a method whose description seems fair. The method should satisfy properties that formalize our notions of fairness. The fairness properties described in the previous section for allocations can be turned into fairness properties for methods.

DEFINITION 2.3 (Unbiased, Strongly Unbiased, Subsidy-Free, Player-Rational, and Coalition-Rational Properties). An allocation method α is *unbiased*, *strongly unbiased*, *subsidy-free*, *player-rational*, or *coalition-rational* on a set Γ of coalition games if $\alpha(N, w)$ is unbiased, strongly unbiased, subsidy-free, player-rational, or coalition-rational, respectively, for all $(N, w) \in \Gamma$.

The egalitarian method is player-rational on superadditive games, and unbiased and strongly unbiased on all games. The egalitarian method is neither subsidy-free nor coalition-rational on superadditive, balanced, convex, or simple games. We would like to describe allocation methods that satisfy fairness properties on as large of a class of games as possible.

The goal of the following sections is to describe several allocation methods, apply them to the previously described examples, and explore their properties. In this section, we describe the fairness properties we will consider.

If we give the players different names, the economic situation is left unchanged and so the allocation should be unchanged except for the renaming. Such an allocation method is anonymous, and an anonymous method is unbiased.

DEFINITION 2.4 (Anonymous Property). If (N, w) is a game, $x \in \mathbb{R}^N$, and $\pi : N \longrightarrow M$ is a bijection, then the game $(\pi(N), \pi w)$ is defined by $(\pi w)(\pi(S)) = w(S)$ and the vector $\pi(x)$ is defined by $\pi(x)_{\pi(i)} = x_i$. An allocation method α is *anonymous* on a set Γ of coalition games if $\alpha(\pi(N), \pi w) = \pi(\alpha(N, w))$ whenever $(N, w) \in \Gamma$, $(\pi(N), \pi w) \in \Gamma$, and $\pi : N \longrightarrow M$ is a bijection.

A change in units (e.g., euros instead of dollars) in the data should result in only a change of units in the allocations.

DEFINITION 2.5 (Proportionate Property). An allocation method α is *proportionate* on a set Γ of coalition games if $\alpha(N,\lambda w)=\lambda\alpha(N,w)$ for all $(N,\lambda w)\in\Gamma$, $(N,w)\in\Gamma$, and $\lambda>0$.

If a game is changed by adding 1 to every coalition containing player i, then the allocation should be unchanged except for giving an additional 1 to player i.

DEFINITION 2.6 (Player-Separable Property). An allocation method α is *player-separable* on a set Γ of coalition games if $\alpha_i(N,w+v)=\alpha_i(N,w)+v(\{i\})$ for all $(N,w+v)\in\Gamma$, $(N,w)\in\Gamma$, additive games (N,v), and $i\in N$.

The egalitarian method is clearly anonymous, proportionate, and player-separable.

We can now present our first allocation method characterization theorem. Notice that the characterization holds on four different classes of games. While we want allocation methods that will satisfy fairness properties on as large of a class of games as possible, when we are characterizing allocation methods, the smaller the class of games, the stronger the result.

THEOREM 2.7. *If α is unbiased and player-separable on (all, superadditive, balanced, or convex) 2-player games, then α is the egalitarian method on (all, superadditive, balanced, or convex) 2-player games.*

PROOF. Suppose (AB,u) is a 2-player game. Define (AB,w) by $w(AB)=u(AB)-u(A)-u(B)$ and $w(A)=w(B)=0$. If u is superadditive, $w(AB)\geq 0$ and so w is convex. Since α is unbiased, $\alpha_A(AB,w)=\alpha_B(AB,w)=\frac{1}{2}w(AB)$. Let (AB,v) be the additive game defined by $v(AB)=u(A)+u(B)$, $v(A)=u(A)$, and $v(B)=u(B)$. Since $u=w+v$ and α is player-separable, $\alpha_i(AB,u)=\alpha_A(AB,w)+v(i)=\frac{1}{2}(u(AB)-u(A)-u(B))+u(i)=\xi_i(AB,u)$ for $i=A,B$. □

In comparing the savings and cost overrun games, we suggested that each player should be allocated at least as much in the savings game as in the cost overrun game. This notion is captured by the following property:

DEFINITION 2.8 (Aggregate-Monotone Property). An allocation method α is *aggregate-monotone* on a set Γ of coalition games if $\alpha_i(N,v)\leq\alpha_i(N,w)$ for all $(N,v)\in\Gamma$, $(N,w)\in\Gamma$, and $i\in N$ satisfying $v(N)\leq w(N)$ and $v(S)=w(S)$ for all $S\subseteq N$ but $S\neq N$.

In addition to player payoffs increasing with increasing $w(N)$, we may wish to require the payoffs to players in any coalition S to increase with increasing $w(S)$. This notion is captured by the following property:

DEFINITION 2.9 (Coalition-Monotone Property). An allocation method α is *coalition-monotone* on a set Γ of coalition games if $\alpha_i(N,v)\leq\alpha_i(N,w)$ for all $(N,v)\in\Gamma$, $(N,w)\in\Gamma$, and $i\in N$ satisfying $v(S)\leq w(S)$ for all $S\subseteq N$ and $v(S)=w(S)$ for all $S\subseteq N\backslash\{i\}$.

Clearly, a coalition-monotone method is aggregate-monotone. The egalitarian method is coalition-monotone. Indeed, $\partial\xi_i(N,w)/\partial w(S)\geq 0$ for all coalitions S containing i.

Felsenthal and Machover **[11]** argue that measuring voting power in superadditive simple games should satisfy their transfer property, which is a strengthening of the coalition-monotone property. Allocation methods that are not coalition-monotone are subject to serious paradoxes when used as measures of *a priori* relative voting power.

This completes our introduction to some fairness properties for allocation methods and our analysis of the egalitarian method. The next three sections describe and analyze many allocation methods. A table summarizing our analysis can be found in section 6. The reader may find it helpful to refer to that table now for the egalitarian method and then while reading the following sections.

3. Weighted Contribution Methods

The egalitarian method is almost never subsidy-free. To define a subsidy-free allocation method, we need to give a payoff of $w(\{i\})$ to a player i whenever $w(S) = w(S\backslash\{i\})+w(\{i\})$ for all coalitions S that contain i. This suggests that after assigning each player their individual gain, we split any additional gains proportional to quantities involving $w(S)-w(S\backslash\{i\})-w(\{i\})$, which can be interpreted as the gain player i brings to coalition S in addition to what player i can gain as an individual.

DEFINITION 3.1 (Marginal and Synergy Contribution). Suppose (N, w) is a game. Player i's *individual gain* is $w(\{i\})$. Player i's *marginal contribution* to the coalition S is the quantity $w(S) - w(S\backslash\{i\})$. Player i's *synergy contribution* to the coalition S is the quantity $w(S) - w(S\backslash\{i\}) - w(\{i\})$.

DEFINITION 3.2 (Synergy, Banzhaf, and Shapley Values). Suppose $a = (a_2, a_3, \ldots, a_n)$ is a non-zero vector of nonnegative real numbers. Player i's *a-weighted sum of synergy contributions* is

$$\bar{\sigma}_i^a(N,w) = \sum_{S\ni i} a_{|S|}\left(w(S) - w(S\backslash\{i\}) - w(\{i\})\right) \tag{3.1}$$

where the notation $S \ni i$ means to sum over all $S \subseteq N$ satisfying $i \in S$. If w is additive, then $\bar{\sigma}_i^a(N,w) = 0$ for all $i \in N$. If w is superadditive, $\bar{\sigma}_i^a(N,w) \geq 0$ for all $i \in N$. The *a-synergy value* σ^a gives player i his individual gain, $w(\{i\})$, plus the remaining available non-individual gains, $w(N) - \sum_{j\in N} w(\{j\})$, in proportion to $\bar{\sigma}_i^a(N,w)$:

$$\sigma_i^a(N,w) = w(\{i\}) + \frac{\bar{\sigma}_i^a(N,w)}{\sum_{j\in N}\bar{\sigma}_j^a(N,w)}\left(w(N) - \sum_{j\in N} w(\{j\})\right) \tag{3.2}$$

if $\sum_{j\in N}\bar{\sigma}_j^a(N,w) \neq 0$ and

$$\sigma_i^a(N,w) = \xi_i(N,w)$$

otherwise. Note that for any $\lambda > 0$, the (λa)-synergy value is the same as the a-synergy value. If $a_s = 1$ for all s, then the a-synergy value is denoted by β and called the (normalized) *Banzhaf value*. If $a_s = (s-1)!(n-s)!$ for all s, then the a-synergy value is denoted by φ and called the *Shapley value*.

The Shapley value was defined in [**29**].and has been studied extensively (e.g., [**3**, chapters 53-54] and [**27**]). The Banzhaf value as defined here was mentioned in [**10**]. The Banzhaf value was originally defined as the average sum of marginals in [**4**] on a subclass of simple games called weighted voting games. This corresponds to (3.1) with $a_s = 1/2^{|N|-1}$ for all s and the synergy contributions replaced with marginal contributions. Of course, such functions on games need not yield payoff vectors that satisfy the efficiency condition. Dubey, Neyman, and Weber **[9]** characterized such functions that are linear, anonymous, and player-separable, and so introduced the idea of weighted sums of marginal values. See Malawski **[20]** for some characterizations of the absolute Banzhaf value and references to earlier characterizations. van den Brink and van der Laan **[33]** characterized the "normalized" Banzhaf value, which yields allocations proportional to the sum of marginals. The normalized Banzhaf value and the Banzhaf value as defined here are identical on simple games and on zero-normalized games. The advantage of the a-synergy values are that they are subsidy-free while the normalized Banzhaf value is not.

The computation of the Banzhaf value for the savings game is summarized in the table. Each upper cell contains the synergy contribution for the player named in the column header to the coalition named in the row header. The last two cells record the sum of the synergy contributions and the Banzhaf value.

	Synergy Contribution			
Player	A	B	C	D
ABCD	140	56	44	32
ABC	108	108	84	
ABD	96	96		72
ACD	84		84	84
AB	24	24		
Sum	452	284	212	188
β_i	55.7	35.0	26.1	23.2

For the cost overrun game, each number in the "ABCD" row and the "Sum" row is reduced by 20, and so the Banzhaf value selects $(49.1, 30.0, 21.8, 19.1)$.

For the Shapley value, before summing the synergy contributions, those in the first row are multiplied by $a_4 = (4-1)!(4-4)! = 6$, those in the second through fourth rows are multiplied by $a_3 = (3-1)!(4-3)! = 2$, and those in the fifth row are multiplied by $a_2 = (2-1)!(4-2)! = 2$. The weighted sums are $(1464, 792, 600, 504)$ and the Shapley value selects $(61, 33, 25, 21)$. For the cost overrun game, each weighted sum is reduced by 120, and so the Shapley value selects $(56, 28, 20, 16)$. Since $56+28+20 < 108$, this example shows that the Shapley value is not coalition-rational on balanced games.

Other a-synergy values can be computed in a similar manner. The next theorem allows us to determine for a game the set of all a-synergy values simultaneously.

THEOREM 3.3. *If (N, w) is superadditive, then the set of $\sigma^a(N, w)$ for all non-zero vectors of nonnegative real numbers $a = (a_2, a_3, \ldots, a_n)$ is the convex hull of $\sigma^{\chi^i}(N, w)$ for $i = 2, 3, \ldots, n$, where χ^i is the ith unit vector*

PROOF. If w is additive, then $\sigma^a(N, w) = \xi(N, w)$ for all a, and so the theorem follows. Now suppose w is not additive. Let

$$d_{i,s} = \sum_{S:i\in S,|S|=s}(w(S) - w(S\backslash\{i\}) - w(\{i\}))$$

and

$$A = w(N) - \sum_{j\in N} w(\{j\}).$$

By the superadditivity condition, $d_{i,s} \geq 0$. Since w is not additive, $d_{i,s} > 0$ for some i and s. We can now let

$$\lambda_s = a_s \left(\sum_{j=1}^{n} d_{j,s}\right) / \left(\sum_{r=2}^{n} a_r \left(\sum_{j=1}^{n} d_{j,r}\right)\right).$$

Clearly, $\lambda_s \geq 0$ and $\sum_{s=2}^{n} \lambda_s = 1$. Now

$$\begin{aligned}
\sigma_i^a(N, w) &= w(i) + A\left(\sum_{s=2}^{n} a_s d_{i,s}\right) / \left(\sum_{j=1}^{n} \sum_{r=2}^{n} a_r d_{j,r}\right) \\
&= \sum_{s=2}^{n} \lambda_s w(i) + \sum_{s=2}^{n} a_s A d_{i,s} / \left(\sum_{r=2}^{n} a_r \left(\sum_{j=1}^{n} d_{j,r}\right)\right) \\
&= \sum_{s=2}^{n} \lambda_s \left(w(i) + A d_{i,s} / \sum_{j=1}^{n} d_{j,s}\right) \\
&= \sum_{s=2}^{n} \lambda_s \sigma_i^{\chi^s}(N, w)
\end{aligned}$$

and the theorem follows. □

For the savings game,

$$\begin{aligned}
\bar{\sigma}^{\chi^2}(N, w) &= (24, 24, 0, 0), \\
\bar{\sigma}^{\chi^3}(N, w) &= (288, 204, 168, 156), \\
\bar{\sigma}^{\chi^4}(N, w) &= (140, 56, 44, 32).
\end{aligned}$$

Hence any a-synergy allocation is a convex combination of $\sigma^{\chi^2}(N, w) = (70, 70, 0, 0)$, $\sigma^{\chi^3}(N, w) \approx (49.4, 35.0, 28.8, 26.8)$, and $\sigma^{\chi^4}(N, w) \approx (72.1, 28.8, 22.6, 16.5)$.

We now examine some properties of a-synergy values. It is easy to verify that a-synergy values are player-rational on superadditive games and are unbiased, anonymous, proportionate, player-separable, and subsidy-free on all games.

The a-synergy values are not coalition-rational on balanced games. Indeed, for an a-synergy value, $a_s > 0$ for some s satisfying $2 \leq s \leq n$. If $a_s > 0$ for some s satisfying $2 \leq s \leq n - 1$, then for a veto power game $\varepsilon = \sigma_n^a(N, v^{1,s}) > 0$ which implies that $\sum_{i=1}^{n-1} \sigma_1^a(N, v^{1,s}) = 1 - \sigma_n^a(N, v^{1,s}) < 1 = v^{1,s}(N \backslash \{n\})$, and so σ^a is not coalition-rational on the balanced game $(N, v^{1,s})$. If $a_s = 0$ for all s satisfying $2 \leq s \leq n - 1$, then $a_n > 0$. Let $w(1234) = 8$, $w(123) = w(124) = w(134) = w(234) = w(12) = 5$, and $w(S) = 0$ otherwise. Then $(3, 3, 1, 1)$ is coalition-rational and $x = \sigma^a(N, w) = (2, 2, 2, 2)$. Hence, $x_1 + x_2 = 4 < w(12)$, and so $\sigma^a(N, w)$ is not coalition-rational on the balanced game (N, w).

On convex games, Shapley [**30**] showed that the Shapley value is coalition-rational and is in the interior of the coalition-rational allocations. This implies that for weights a sufficiently close to the Shapley weights, the a-synergy value is also coalition-rational on convex games.

On the other hand, the Banzhaf value is not coalition-rational on convex games. Indeed, if $w = 31u^{12} + 93u^{23456}$, then w is convex and

$$\begin{aligned}
\bar{\beta}(N, w) &= 31\bar{\beta}(N, u^{12}) + 93\bar{\beta}(N, u^{23456}) \\
&= 31(2^4, 2^4, 0, 0, 0, 0) + 93(0, 2, 2, 2, 2, 2) \\
&= (496, 682, 186, 186, 186, 186),
\end{aligned}$$

and so $\beta(N, w) = (32, 44, 12, 12, 12, 12)$, which implies that $\sum_{i=2}^{6} \beta_i(N, w) = 92 < 93 = w(23456)$.

Not all a-synergy values are aggregate-monotone. Indeed, let $a = \chi^n$, and consider increasing the worth of the grand coalition from a veto power game, $w = v^{1,n-1} + \varepsilon u^N$. Then $\bar{\sigma}^a(N,w) = (1+\varepsilon, \varepsilon, \ldots, \varepsilon)$, and $\sigma_1^a(N,w) = (1+\varepsilon)^2/(1+n\varepsilon)$. Hence, $(\partial\sigma_1^a/\partial\varepsilon)_{\varepsilon=0} = 2-n < 0$, and so σ^a is not aggregate-monotone.

The Banzhaf value is aggregate-monotone on superadditive games. Because the Banzhaf value is player-separable, it is sufficient to show $\partial\beta_i/\partial w(N)$ at superadditive games (N,w) satisfying $w(S) = 0$ if $|S| = 1$. This simplifies

$$\bar{\beta}_i(N,w) = \sum_{S \ni i} (w(S) - w(S\backslash\{i\})) = \sum_{S \ni i} w(S) - \sum_{S \not\ni i} w(S)$$

(by separating the individual terms into separate sums),

$$\sum_{j\in N} \bar{\beta}_j(N,w) = \sum_{S\subseteq N} (2\,|S| - |N|)w(S)$$

(by interchanging summations over j and S), and

$$\beta_i(N,w) = \frac{\bar{\beta}_i(N,w)}{\sum_{j\in N} \bar{\beta}_j(N,w)} w\,(N)\,.$$

Using the quotient and product derivative rules, the numerator of $\partial\beta_i/\partial w(N)$ is

$$\begin{aligned}
&\sum_{S\subseteq N} (2\,|S| - |N|)w(S)\left(w(N) + \bar{\beta}_i(N,w)\right) - \bar{\beta}_i(N,w)w(N)\,|N| \\
&= w(N) \sum_{S\subseteq N} (2\,|S| - |N|)w(S) + \bar{\beta}_i(N,w) \sum_{S\neq N} (2\,|S| - |N|)w(S) \\
&= w(N) \sum_{j\in N} \bar{\beta}_j(N,w) + \bar{\beta}_i(N,w) \textstyle\sum_{s=1}^{\lfloor |N|/2 \rfloor} (|N| - 2s) \left(\displaystyle\sum_{\substack{S\subseteq N:\\ |S|=n-s}} w(S) - \sum_{\substack{S\subseteq N:\\ |S|=s}} w(S) \right)
\end{aligned}$$

which is nonnegative because (N,w) is superadditive.

The Banzhaf value is neither coalition-monotone nor strongly unbiased on convex games. Indeed, let $v(1234) = 32$, $v(S) = 8$ if $|S| = 3$, $v(S) = 4$ if $|S| = 2$, and $v(S) = 0$ if $|S| = 1$. Let $w(1) = 4$ and $w(S) = v(S)$ otherwise. It is easily verified that v and w are convex, $\beta(v) = (8,8,8,8)$, and $\beta(w) = (\frac{146}{19}, \frac{154}{19}, \frac{154}{19}, \frac{154}{19})$. Since $\beta_1(v) > \beta_1(w)$, the Banzhaf value is not coalition-monotone. Since $w(R\cup\{1\}) \geq w(R\cup\{2\})$ for all $R \subseteq \{3,4\}$ and $\beta_1(w) < \beta_2(w)$, the Banzhaf value is not strongly unbiased.

The Banzhaf value is not coalition-monotone on simple games. Felsenthal and Machover [11] provided this counterexample: let $N = \{1,2,3,4,5\}$, $v = u^{12}$, and w be the same as v except that $w(1345) = 1$. Then $\beta(N,v) = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$ and $\beta(N,w) = (\frac{9}{19}, \frac{7}{19}, \frac{1}{19}, \frac{1}{19}, \frac{1}{19})$. The non-monotoncity is provided by $\beta_1(N,v) > \beta_1(N,w)$.

All a-synergy values are strongly unbiased on zero-normalized games and on superadditive simple games. Indeed, suppose $w(S\cup\{i\}) \leq w(S\cup\{j\})$ for all $S \subseteq$

$N\backslash\{i,j\}$. If (N,w) is zero-normalized, then

$$\begin{aligned}\bar{\sigma}_i^a(N,w) &= \sum\nolimits_{R\subseteq N\backslash\{i,j\}} a_{|R|+1}(w(R\cup\{i\})-w(R)) \\ &+\sum\nolimits_{R\subseteq N\backslash\{i,j\}} a_{|R|+2}(w(R\cup\{i,j\})-w(R\cup\{j\})) \\ &\le \sum\nolimits_{R\subseteq N\backslash\{i,j\}} a_{|R|+1}(w(R\cup\{j\})-w(R)) \\ &+\sum\nolimits_{R\subseteq N\backslash\{i,j\}} a_{|R|+2}(w(R\cup\{i,j\})-w(R\cup\{i\})) \\ &= \bar{\sigma}_j^a(N,w).\end{aligned}$$

If (N,w) is superadditive and simple but not zero-normalized, then $w=u^{\{k\}}$ for some $k\in N\backslash\{i\}$, and $\bar{\sigma}^a(N,w)=\chi^k$, which implies $\bar{\sigma}_i^a(N,w)\le\bar{\sigma}_j^a(N,w)$.

In a superadditive simple game, we say that a player i *swings* for coalition S if S is winning and $S\backslash\{i\}$ is losing. For a superadditive simple game without a dictator, the sum in the absolute Banzhaf value formula is the number of times the player swings. For the federal law game, a Representative swings whenever in a coalition of (1) 217 other Representatives, at least 50 Senators,and the President, or (2) 289 other Representatives and at least 67 other Senators. So, a Representative has the following number of swings:

$$r=\binom{434}{217}\sum_{s=50}^{100}\binom{100}{s}+\binom{434}{289}\sum_{s=67}^{100}\binom{100}{s}$$

A Senator swings whenever in a coalition of (1) at least 218 Representatives, 49 other Senators, and the President, or (2) at least 290 Representatives and 66 other Senators. So, a Senator has the following number of swings:

$$s=\binom{99}{49}\sum_{r=218}^{435}\binom{435}{r}+\binom{99}{66}\sum_{r=290}^{435}\binom{435}{r}$$

The President swings whenever in a coalition of (1) at least 218 Representatives and between 50 and 66 Senators, or (2) between 218 and 289 Representatives and at least 67 Senators. So, the President has the following number of swings:

$$p=\left(\sum_{r=218}^{435}\binom{435}{r}\right)\left(\sum_{s=50}^{66}\binom{100}{s}\right)+\left(\sum_{r=218}^{289}\binom{435}{r}\right)\left(\sum_{s=67}^{100}\binom{100}{s}\right)$$

The Banzhaf payoffs can be obtained by dividing the above quantities by $435r+100s+p$. The Banzhaf payoff for a Representative, Senator, and President are 0.00153, 0.00295, and 0.03996, respectively.

4. Shapley Value

In the previous section, we defined the Shapley value φ on n-player games as the a-synergy value with $a_s=(s-1)!(n-s)!$. An equivalent description is that the Shapley value gives each player that player's marginal contribution to each coalition, averaged over all player orders.

THEOREM 4.1. *The* Shapley value φ *is given by*

$$\varphi_i(N,w)=\frac{1}{|N|!}\sum_{\pi}\left(w\left(S^{\pi,i}\right)-w\left(S^{\pi,i}\backslash\{i\}\right)\right) \tag{4.1}$$

where the sum is over all player orders, i.e., one-to-one and onto functions $\pi : N \to \{1, 2, \ldots, |N|\}$, *and* $S^{\pi,i} = \{j \in N : \pi(j) \leq \pi(i)\}$ *is the coalition of player* i *and the players that come before* i *in the order* π. *An equivalent formula is*

$$\varphi_i(N, w) = \sum_{S \subseteq N} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!} \left(w(S) - w(S \setminus \{i\})\right). \tag{4.2}$$

Proof. By definition (3.1),

$$\bar{\varphi}_i(N, w) = \sum_{S \ni i} (|S|-1)!(n-|S|)!\,(w(S) - w(S\setminus\{i\}) - w(\{i\}))$$

where $n = |N|$. Since

$$\begin{aligned}\sum_{S: i \in S} (|S|-1)!(|N|-|S|)! &= \sum_{s=1}^{n} \binom{n-1}{s-1} (s-1)!(n-s)! \\ &= \sum_{s=1}^{n} (n-1)! = n!,\end{aligned}$$

it follows that

$$\bar{\varphi}_i(N, w) = \sum_{S \ni i} (|S|-1)!(n-|S|)!\,(w(S) - w(S\setminus\{i\})) - n!w(\{i\}).$$

By interchanging the summations over i and S, we obtain

$$\sum_{i \in N} \bar{\varphi}_i(N, w) = \sum_{S \subseteq N} \sum_{i \in S} (|S|-1)!(n-|S|)!\,(w(S) - w(S\setminus\{i\})) - n! \sum_{i \in N} w(\{i\})$$

For the double sum and a fixed coalition S, the worth $w(S)$ appears as a positive term with coefficient $(|S|-1)!(n-|S|)!$ for each $i \in S$, and the worth $w(S)$ appears as a negative term if $S \neq N$ with coefficient $|S|!(n-|S|-1)!$ for each $i \in N\setminus S$. After simplifying the double sum, the coefficient for $w(N)$ is

$$(n-1)!(n-n)!n = n!,$$

and if $S \neq N$, the coefficient for $w(S)$ is

$$(|S|-1)!(n-|S|)!\,|S| - |S|!(n-|S|-1)!(n-|S|) = 0.$$

Hence,

$$\sum_{i \in N} \bar{\varphi}_i(N, w) = n! \left(w(N) - \sum_{i \in N} w(\{i\}) \right)$$

Plugging into (3.2) and simplifying, we obtain (4.2). We can also obtain (4.2) from (4.1) by counting the number of orders in which player i comes after the players in $S\setminus\{i\}$ and before the players in $N\setminus S$. □

The computation of the Shapley value for the savings game is summarized in the table. Each row corresponds to one of the 24 possible player orders. In the eighth row (in boldface), player B decides to cooperate first and is given the marginal contribution $w(B) - w(\varnothing) = 0 - 0 = 0$, player A decides to cooperate second and is given the marginal contribution $w(BA) - w(B) = 24 - 0 = 24$, player D decides to cooperate third and is given the marginal contribution $w(BAD) - w(BA) = 96 - 24 = 72$, and player C decides to cooperate fourth and is given the marginal contribution $w(BADC) - w(BAD) = 140 - 96 = 44$. The other rows are calculated in an analogous fashion. Finally, the marginal contributions are averaged to obtain the Shapley allocation (61, 33, 25, 21).

Order				Marginal Contribution			
$\pi(A)$	$\pi(B)$	$\pi(C)$	$\pi(D)$	A	B	C	D
1	2	3	4	0	24	84	32
1	2	4	3	0	24	44	72
1	3	2	4	0	108	0	32
1	4	2	3	0	56	0	84
1	3	4	2	0	96	44	0
1	4	3	2	0	56	84	0
2	1	3	4	24	0	84	32
2	**1**	**4**	**3**	**24**	**0**	**44**	**72**
3	1	2	4	108	0	0	32
4	1	2	3	140	0	0	0
3	1	4	2	96	0	44	0
4	1	3	2	140	0	0	0
2	3	1	4	0	108	0	32
2	4	1	3	0	56	0	84
3	2	1	4	108	0	0	32
4	2	1	3	140	0	0	0
3	4	1	2	84	56	0	0
4	3	1	2	140	0	0	0
2	3	4	1	0	96	44	0
2	4	3	1	0	56	84	0
3	2	4	1	96	0	44	0
4	2	3	1	140	0	0	0
3	4	2	1	84	56	0	0
4	3	2	1	140	0	0	0
				61	33	25	21

For the cost overrun game, the marginal contributions in the table stay the same except that for each order, the fourth player's marginal contribution is reduced by 20. Since each player is fourth in one-fourth of the orders, each player's average marginal contribution is reduced by $20/4 = 5$. Hence, the Shapley allocation for the cost overrun game is (56, 28, 20, 16), which is not coalition-rational (consider coalition ABC).

Using (4.2), it is easy to verify that the Shapley value is strongly unbiased and coalition-monotone on all games.

For a superadditive simple game and each player order, there is exactly one marginal contribution of 1, corresponding to the *pivotal* player whose addition changes a losing coalition to a winning coalition, and all other marginal contributions are 0. Hence for superadditive simple games, the Shapley payoff to a player is the fraction of times that player is pivotal among all player orders.

For the federal law game, a Representative is pivotal whenever in a coalition of (1) 217 other Representatives, at least 50 Senators, and the President, or (2) 289 other Representatives and at least 67 other Senators. So, the Shapley value for a

Representative is

$$\frac{1}{536!}\left(\binom{434}{217}\sum_{s=50}^{100}\binom{100}{s}(218+s)!(317-s)!+\right.$$
$$\left.\binom{434}{289}\sum_{s=67}^{100}\binom{100}{s}(289+s)!(246-s)!\right) \approx 0.0010069$$

A Senator is pivotal whenever in a coalition of (1) at least 218 Representatives, 49 other Senators, and the President, or (2) at least 290 Representatives and 66 other Senators. So, the Shapley value for a Senator is

$$\frac{1}{536!}\left(\binom{99}{49}\sum_{r=218}^{435}\binom{435}{r}(50+r)!(485-r)!+\right.$$
$$\left.\binom{99}{66}\sum_{r=290}^{435}\binom{435}{r}(66+r)!(469-r)!\right) \approx 0.0039658$$

Since the Shapley value selects allocations, the Shapley value for the President is

$$1-(435)(0.0010069)-(100)(0.0039658) \approx 0.16542.$$

It is interesting to speculate whether the President has roughly 4% of the voting power related to the passage of federal laws, as suggested by the Banzhaf value, or more than 16%, as suggested by the Shapley value. Brams, Affuso, and Kilgour **[6]** argue that informal analysis and empirical data suggest that the President's power should be comparable to the power of at least one half of the Representatives or Senators combined. As this is more than suggested by either the Banzhaf or Shapley values, Brams, Affuso, and Kilgour advocate the use of a method introduced by Johnston **[16]**, which suggests that the President has 77% of the power. On the other hand, Johnston's method is not coalition-monotone, and Felsenthal and Machover **[11]** argue that any allocation method for voting power must be coalition-monotone. There are a-synergy values that assign the President large powers, e.g., the χ^{356}-synergy value assigns the President 99.6% of the power. There is some regularity to Presidential power assigned by χ^r-synergy values, e.g., President power monotonically increases from 9.4% to 99.6% with r increasing from 280 to 356 (one less than needed for a veto override). It is unclear why it would be appropriate to choose the a_r weights with r near but below 356 to be greatest. Perhaps further investigation of which a-synergy values are coalition-monotone would be helpful.

For the veto power games, it is straightforward to calculate

$$\bar{\beta}_i(N,v^{1,r}) = \begin{cases} \sum_{s=r}^{n}\binom{n-1}{s-1}, & \text{if } i=1 \\ \binom{n-2}{r-2}, & \text{if } i>1 \end{cases}$$

and

$$\varphi_i(N,v^{1,r}) = \begin{cases} 1-\frac{r-1}{n}, & \text{if } i=1 \\ \frac{r-1}{n(n-1)}, & \text{if } i>1 \end{cases}$$

The table provides a numerical comparison among the Banzhaf, Shapley, and coalition-rational values on small veto power games. The key observation is that a-synergy values detect differences in the powers of the players in different veto power games while a coalition-rational allocation method would not.

n	3	4	4	5	5	5
r	2	3	2	4	3	2
$\beta_1(N, v^{1,r})$	0.60	0.40	0.70	0.29	0.48	0.79
$\varphi_1(N, v^{1,r})$	0.67	0.50	0.75	0.40	0.60	0.80
coalition-rational	1.00	1.00	1.00	1.00	1.00	1.00

5. Weighted Nucleoli

A coalition-rational allocation x for the game (N, w) satisfies $\sum_{i \in S} x_i \geq w(S)$ for all $S \subseteq N$. This motivates us to find an allocation that minimizes the quantities $w(S) - \sum_{i \in S} x_i$. Of course, we cannot simultaneously minimize all of these quantities. Instead, we will minimize the maximum of these quantities after multiplying each quantity by some weighting factor.

DEFINITION 5.1 (Nucleolus). Suppose (N, w) is a coalition game and $a = (a_1, a_2, \ldots)$ is a sequence of positive real numbers. The *a-excess* of coalition S at an allocation x is the quantity

$$e^a(S, x) = a_{|S|}\left(w(S) - \sum_{i \in S} x_i\right).$$

The *a-excess vector* at an allocation x, denoted $e^a(x)$, is the vector of numbers $e^a(S, x)$ for $S \subseteq N$ ordered from largest to smallest. We order excess vectors lexicographically, that is, $e(x) <_{lex} e(y)$ if there is a positive integer k for which $e_j(x) = e_j(y)$ for $j < k$, and $e_k(x) < e_k(y)$. The *a-nucleolus* $\nu^a(N, w)$ for (N, w) is the player-rational allocation whose a-excess vector is the lexicographic minimum. The *a-prenucleolus* for (N, w) is the allocation (not necessarily player-rational) whose a-excess vector is the lexicographic minimum. If a is a constant sequence, we obtain the *nucleolus* $\nu = \nu^a$. If $a_k = 1/k$, we obtain the *per capita nucleolus* $\nu^{PC} = \nu^a$.

The nucleolus was defined by Schmedler **[28]**, and the per capita nucleolus was defined by Grotte **[12]**. Wallmeier **[34]** defined weighted versions but limited his investigation to nonincreasing sequences a. Derks and Haller **[7]** considered a more general class of weighted nucleoli in which the weights depend on n and S, instead of only the cardinality of S. Our smaller class was chosen so that the values would be anonymous.

The excess $e^a(S, x)$ is a measure of coalition S's dissatisfaction with the allocation x. The allocation x is coalition-rational if and only if all excesses at x are nonpositive. By minimizing the maximum excess, the a-nucleolus will be coalition-rational if coalition-rational allocations exist. Since the veto power game $(N, v^{i,r})$ has the unique coalition-rational allocation χ^i, it follows that $\nu^a(N, v^{i,r}) = \chi^i$ for arbitrary a.

It is not immediately clear from its definition whether an a-nucleolus exists and is unique. Proofs appear in **[7]**, **[28]**, and **[34]**. The approach for the existence proof is to note that the set X_0 of player-rational allocations is nonempty and compact. We now inductively construct subsets of X_0 and subcollections of $\mathcal{C}_0$, the collection of all coalitions: Given a nonempty and compact set of allocations X_k

and a non-empty collection $\mathcal{C}_k$ of subsets of N, the function

$$f_k(x) = \max_{S \in \mathcal{C}_k} e^a(S, x)$$

is a continuous function on X_k, and so

$$b_{k+1} = \min_{x \in X_k} f_k(x)$$

exists, the set

$$X_{k+1} = \{x \in X_k : f_k(x) = b_k\}$$

is nonempty and compact,

$$\mathcal{B}_{k+1} = \{S \in \mathcal{C}_k : e^a(S, x) = b_k \text{ for all } x \in X_{k+1}\}$$

is nonempty, and the collection

$$\mathcal{C}_{k+1} = \mathcal{C}_k \backslash \mathcal{B}_{k+1}$$

is a strict subset of $\mathcal{C}_k$. Eventually, $\mathcal{C}_{k+1} = \varnothing$, and the elements of X_{k+1} are allocations whose a-excess vector are the lexicographic minimums among allocations in X_0. Notice that each b_k and X_k can be found by solving a linear program.

By definition, every a-nucleolus is player-rational on all games. It is easy to verify that every a-nucleolus is unbiased, anonymous, proportionate, and player-separable on all games. Before we consider other properties, we will determine the nucleolus and per capita nucleolus for our examples. Instead of solving linear programs, we will propose an allocation and provide a verification that the proposal is the a-nucleolus.

For the savings game, the table shows the excesses for the allocations $x = (74, 28, 22, 16)$, $y = (84, 20, 20, 16)$, and $z = (68, 36, 20, 16)$.

S	$e(S,x)$	$e(S,y)$	$e(S,z)$
ABC	-16	-16	-16
ABD	-22	-24	-24
ACD	-28	-20	-36
BCD	-66	-72	-56
AB	-78	-80	-80
AC	-96	-88	-104
AD	-90	-84	-100
BC	-50	-56	-40
BD	-44	-52	-36
CD	-38	-36	-36
A	-74	-68	-84
B	-28	-36	-20
C	-22	-20	-20
D	-16	-16	-16

The first three components of $e(x)$ are -16, -16, and -22, and the first three components of $e(y)$ are -16, -16, and -20; hence, $e(x) <_{lex} e(y)$. The vectors $e(y)$ and $e(z)$ agree on the first 7 components and $e_8(y) = -52 < -36 = e_8(z)$; hence, $e(y) <_{lex} e(z)$. This shows that neither y nor z is the nucleolus.

The allocation $x = (74, 28, 22, 16)$ is the nucleolus for the savings game. Indeed, suppose p is an allocation satisfying $e(p) \leq_{lex} e(x)$. First, since $e_1(x) = -16$, it follows that $e(S,p) \leq -16$ for all $S \neq \varnothing, N$. In particular,

$$-16 \geq e(ABC,p) = 108 - p_A - p_B - p_C$$
$$-16 \geq e(D,p) = -p_D.$$

Summing these two inequalities and using the efficiency condition, we obtain

$$\begin{aligned} -32 &\geq 108 - p_A - p_B - p_C - p_D \\ &= 108 - 140 = -32. \end{aligned}$$

So, the inequalities must be equalities, and we have $p_D = 16$ and $e(ABC,p) = e(D,p) = -16$. Second, since $e_i(p) = e_i(x)$ for $i \leq 2$, and $e(p) \leq_{lex} e(x)$, it follows that $e(S,p) \leq e_3(x) = -22$ for all $S \neq \varnothing, N, ABC, D$. In particular,

$$-22 \geq e(ABD,p) = 96 - p_A - p_B - p_D$$
$$-22 \geq e(C,p) = -p_C.$$

Summing these two inequalities and using the efficiency condition, we obtain

$$\begin{aligned} -44 &\geq 96 - p_A - p_B - p_C - p_D \\ &= 96 - 140 = -44. \end{aligned}$$

So, the inequalities must be equalities, and we have $p_C = 22$ and $e(ABD,p) = e(C,p) = -22$. Third, since $e_i(p) = e_i(x)$ for $i \leq 4$, and $e(p) \leq_{lex} e(x)$, it follows that $e(S,p) \leq e_5(x) = -28$ for all $S \neq \varnothing, N, ABC, D, ABD, C$. In particular,

$$-28 \geq e(ACD,p) = 84 - p_A - p_B - p_D$$
$$-28 \geq e(B,p) = -p_B.$$

Summing these two inequalities and using the efficiency condition, we obtain

$$\begin{aligned} -56 &\geq 84 - p_A - p_B - p_C - p_D \\ &= 84 - 140 = -56. \end{aligned}$$

So, the inequalities must be equalities, and we have $p_B = 28$. Fourth, using the efficiency condition, $p_A = 140 - p_B - p_C - p_D = 74$, and $p = x$.

The allocation $x = (84, 18, 12, 6)$ is the nucleolus for the cost overrun game. Indeed, the reader can readily verify that the first six components of $e(x)$ are again the six excesses $e(ABC,x) = e(D,x)$, $e(ABD,x) = e(C,x)$, and $e(ACD,x) = e(B,x)$. The argument of the previous paragraph, with $w(ABCD)$ changed from 140 to 120, shows that x is the nucleolus for the cost overrun game.

Since $\nu_1(w^{\text{savings}}) < \nu_1(w^{\text{cost overrun}})$, the nucleolus is not aggregate-monotone on balanced games.

The per capita nucleolus for the savings game is $x = (101.6, 17.6, 12.8, 8)$. Indeed, the first seven components of $e^{PC}(x)$ are

$$e^{PC}(ABC,x) = e^{PC}(D,x) = -8$$
$$e^{PC}(ABD,x) = e^{PC}(CD,x) = -10.4$$
$$e^{PC}(ACD,x) = e^{PC}(BD,x) = e^{PC}(C,x) = -12.8.$$

Suppose p is an allocation satisfying $e^{PC}(p) \leq_{lex} e^{PC}(x)$. First, since $e_1^{PC}(x) = -8$, it follows that $e(S,p) \leq -8$ for all $S \neq \varnothing, N$. In particular,

$$-8 \geq e^{PC}(ABC,p) = (108 - p_A - p_B - p_C)/3$$
$$-8 \geq e^{PC}(D,p) = -p_D.$$

Adding three times the first inequality to the second inequality and using the efficiency condition, we obtain

$$-32 \geq 108 - p_A - p_B - p_C - p_D = -32.$$

So, the inequalities must be equalities, and we have $p_D = 8$ and $e^{PC}(ABC, p) = e^{PC}(D, p) = -8$. Second, since $e_i^{PC}(p) = e_i^{PC}(x)$ for $i \leq 2$, and $e^{PC}(p) \leq_{lex} e^{PC}(x)$, it follows that $e^{PC}(S, p) \leq e_3^{PC}(x) = -10.4$ for all $S \neq \varnothing, N, ABC, D$. In particular,

$$-10.4 \geq e^{PC}(ABD, p) = (96 - p_A - p_B - p_D)/3$$
$$-10.4 \geq e^{PC}(CD, p) = (-p_C - p_D)/2.$$

Adding three times the first inequality to two times the second inequality and using the efficiency condition and $p_D = 8$, we obtain

$$\begin{aligned} -52 &\geq 96 - p_A - p_B - p_C - 2p_D \\ &= 96 - 140 - 8 = -52. \end{aligned}$$

So, the inequalities must be equalities, and we have $p_C = 12.8$ and $e^{PC}(ABD, p) = e^{PC}(C, p) = -10.4$. Third, since $e_i^{PC}(p) = e_i^{PC}(x)$ for $i \leq 4$, and $e^{PC}(p) \leq_{lex} e^{PC}(x)$, it follows that $e^{PC}(S, p) \leq e_5^{PC}(x) = -12.8$ for all $S \neq \varnothing, N, ABC, D, ABD, CD$. In particular,

$$-12.8 \geq e^{PC}(ACD, p) = (84 - p_A - p_C - p_D)/3$$
$$-12.8 \geq e^{PC}(BD, p) = (-p_B - p_D)/2.$$

Adding three times the first inequality to two times the second inequality and using the efficiency condition and $p_D = 8$, we obtain

$$\begin{aligned} -64 &\geq 84 - p_A - p_B - p_C - 2p_D \\ &= 84 - 140 - 8 = -64. \end{aligned}$$

So, the inequalities must be equalities, and we have $p_B = 17.6$. Fourth, using the efficiency condition, $p_A = 140 - p_B - p_C - p_D = 101.6$, and $p = x$.

The per capita nucleolus for the savings game is $x = (96.6, 12.6, 7.8, 3)$. Indeed, the first six components of $e^{PC}(x)$ are the six excesses $e^{PC}(ABC, x) = e^{PC}(D, x)$, $e^{PC}(ABD, x) = e^{PC}(CD, x)$, and $e^{PC}(ACD, x) = e^{PC}(BD, x)$. The argument of the previous paragraph, with $w(ABCD)$ changed from 140 to 120, shows that x is the per capita nucleolus for the cost overrun game.

The approach we have been using to verify that a proposed allocation is the a-nucleolus for a game was first generalized by Kohlberg [**17**] to the nucleolus and then by Potters and Tijs [**26**] for the a-nucleolus. They showed that an allocation is the a-nucleolus if and only if each collection of coalitions whose excess is greater than some number satisfies a combinatorial balancing condition.

We now compute the nucleolus and per capita nucleolus for the federal law game. Since the a-nucleolus is unbiased, the payoff to each representative is the same number r, the payoff to each senator is the same number s, and the payoff to the president is p. Using the efficiency condition, $435r + 100s + p = 1$. Since the a-nucleolus is player-rational, the winning coalition excesses are nonnegative and the losing coalition excesses are nonpositive. If the weights a are nonincreasing (as is the case for the nucleolus and per capita nucleolus), it follows that the maximum excesses will occur with the minimal winning coalitions..The excess for the regular minimal winning coalitions is $a_{269}(1 - 218r - 50s - p) = a_{269}(217r + 50s)$, and the excess for the veto override minimal winning coalitions is $a_{357}(1 - 290r - 67s)$. In order to minimize the maximum excess, we must solve

$$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq a_{269}(217r + 50s) \\ & z \geq a_{357}(1 - 290r - 67s) \\ & 435r + 100s + p = 1 \\ & r \geq 0, s \geq 0, p \geq 0 \end{array}$$

For the weights a of interest,

$$\begin{aligned} r &= 0 \\ s &= a_{357}/(50a_{269} + 67a_{357}) \\ p &= (50a_{269} - 33a_{357})/(50a_{269} + 67a_{357}). \end{aligned}$$

For the nucleolus, $s = 1/117 \approx 0.0085$ and $p = 17/117 \approx 0.1453$. For the per capita nucleolus, $s \approx 0.0075$ and $p \approx 0.2501$. Just as for the veto power games, the a-nucleoli allocate more to the strongest player in the federal law game than the a-synergy values. It is interesting to note that for large a_{269} in comparison to a_{357} the a-nucleolus allocates almost all power to the President. This suggests that if larger coalitions are considered far less important than smaller coalitions (perhaps because smaller coalitions are far more likely to form), then Presidential power is greater.

Every a-nucleolus is subsidy-free on all games. Indeed, suppose (N, w) is a game in which player k is a dummy and x is a player-rational allocation for which $x_k \neq w(\{k\})$. Since x is player-rational, $x_k > w(\{k\})$. We will show that x is not the a-nucleolus. Since k is a dummy, for all coalitions S containing k, it follows that

$$\begin{aligned} e^a(S\backslash\{k\}, x) - e^a(S, x) &= -w(S\backslash\{k\}) - w(S) + x_k \\ &> -w(S\backslash\{k\}) - w(S) + w(\{k\}) \\ &= 0. \end{aligned}$$

By taking a small amount from k and giving it to other players, we can obtain a new player-rational allocation y satisfying $e_1^a(y) < e_1^a(x)$, and so x is not the a-nucleolus.

Every a-nucleolus is strongly unbiased on all games. Indeed, suppose (N, w) is a game in which player i is weaker than player j, and x is a player-rational allocation for which $x_i > x_j$. We will show that x is not the a-nucleolus. For all $R \subseteq N\backslash\{i, j\}$, our supposition implies

$$\begin{aligned} e^a(R \cup \{j\}, x) - e^a(R \cup \{i\}, x) &= w(R \cup \{j\}) - w(R \cup \{i\}) - x_j + x_i \\ &> 0. \end{aligned}$$

Hence, by taking a small amount from i and giving it to j, we can obtain a new player-rational allocation y satisfying $e^a(R \cup \{j\}, x) > e^a(R \cup \{j\}, y) > e^a(R \cup \{i\}, y) > e^a(R \cup \{i\}, x)$ for all $R \subseteq N\backslash\{i, j\}$ and $e^a(S, x) = e^a(S, x)$ for all $S \subseteq N$ satisfying $S \cap \{i, j\} = \{i, j\}$ or $\varnothing$. Thus, $e^a(y) <_{lex} e^a(x)$, and so x is not the a-nucleolus.

The nucleolus is not aggregate-monotone on convex games. Hokari [**13**] provided the following counterexample. Let $N = \{1, 2, 3, 4\}$ and $v(1234) = 10$,

$v(123) = 4$, $v(124) = v(134) = v(234) = 6$, $v(12) = v(14) = v(23) = v(24) = v(34) = 2$, and $v(S) = 0$ otherwise. It can be verified that (N, v) and $(N, v + 2u^{1234})$ are convex games, $\nu(N, v) = (2, 2, 2, 4)$, and $\nu(N, v + 2u^{1234}) = (3, 3, 3, 3)$.

The per capita nucleolus is aggregate-monotone on all games. Young, Okada, and Hashimoto **[38]** proved this, and we sketch the proof for the per capita *prenucleolus*. Suppose (N, v) and (N, w) are games satisfying $v(N) \le w(N)$ and $v(S) = w(S)$ for all $S \ne N$. Let f be defined by $f_i(x) = x_i + \varepsilon$, where $\varepsilon = (w(N) - v(N))/|N|$. Clearly, f is a bijection between the allocations of (N, v) and the allocations of (N, w). Also,

$$\begin{aligned}(v(S) - \sum_{i \in S} x_i)/|S| &= (w(S) - \sum_{i \in S}(f_i(x) - \varepsilon))/|S| \\ &= (w(S) - \sum_{i \in S} f_i(x))/|S| + \varepsilon,\end{aligned}$$

that is, in going from x to $f(x)$, the excesses all increase by ε. Thus, if x is the $\le_{lex}$ minimum on the set of allocations of (N, v), then $f(x)$ is the $\le_{lex}$ minimum on the set of allocations of (N, w). That is, if x is the per capita prenucleolus, then $f(x)$ is the per capita prenucleolus.

Theorem 6.1 will show that no a-nucleolus is coalition-monotone on balanced games.

If the weights a are nonincreasing (which includes the nucleolus and per capita nucleolus), then the a-nucleolus is not coalition-monotone on simple games. Indeed, let (N, v) be the five-player superadditive simple game with minimal winning coalitions 134, 135, 145, 234, 235, and 245. Then $\nu^a(N, v) = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ because the corresponding excess for each minimal winning coalitions is $\frac{1}{3}a_3$, the only other positive excess is $\frac{1}{3}a_4 \le \frac{1}{3}a_3$ for 1234, 1235, and 1245, and if x is any other player-rational allocation satisfying $e^a(x) \le_{\text{lex}} e^a(0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then $e^a(S, x) \le \frac{1}{3}a_3$ for all minimal winning coalitions S, $a_3 x_1 \ge 0$, and $a_3 x_2 \ge 0$, which when summed yield $2a_3 \le 2a_3$ implying that these inequalities hold with equality yielding $x = (0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Let (N, w) be the five-player superadditive simple game with minimal winning coalitions 134, 135, 145, 234, 235, 245, and 123. Then $\nu^a(N, w) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ because the corresponding excess for each minimal winning coalitions is $\frac{2}{5}a_3$, the only other positive excess is $\frac{1}{5}a_4 < \frac{2}{5}a_3$ for the four-player coalitions, and if x is any other player-rational allocation satisfying $e^a(x) \le_{\text{lex}} e^a(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, then $e^a(S, x) \le \frac{2}{5}a_3$ for $S = 134$, 145, 235, 245, and 123, which when summed yield $2a_3 \le 2a_3$ implying that these inequalities hold with equality yielding $x = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. Since $\nu^a_3(N, v) > \nu^a_3(N, w)$, ν^a is not coalition-monotone.

It is an open question whether some a-nucleoli are coalition-monotone on convex or simple games. It would also be interesting to characterize which a-nucleoli are aggregate-monotone on all games.

6. Method Characterizations

In the following table, we state the maximal class(es) of games (among all, superadditive, balanced, convex, zero-normalized, and/or simple) on which each method satisfies each property. There are question marks when it is known that the property does not hold on any larger classes but it is not known whether the property holds on the stated class. All of these methods are unbiased, anonymous, proportionate, and player-separable on all games.

	ξ	β	φ	υ	ν^{PC}
Subsidy-Free	None	All	All	All	All
Strongly Unbiased	All	Zero ∪ Super Simple	All	All	All
Player-Rational	Super	Super	Super	All	All
Coalition-Rational	None	None	Convex	Balanced	Balanced
Aggregate-Monotone	All	Super	All	None	All
Coalition-Monotone	All	Zero ∩ Convex?	All	None	Convex?

Each method has its positive and negative features. In particular, the Shapley value and the per capita nucleolus satisfy most of the properties. However, the Shapley value is not coalition-rational on balanced games and the per capita nucleolus is not coalition-monotone on balanced games. Since both properties are desirable, it is natural to ask whether a method exists that has both properties.

THEOREM 6.1. *There is no coalition-rational and coalition-monotone allocation method for balanced games with four or more players .*

This impossibility was shown for five or more players by Young [**35**]. The proof below is by Housman and Clark [**15**].

PROOF. Suppose α is a coalition-rational and coalition-monotone allocation method for balanced games. Let $N = \{1,2,3,4\}$ and $w(N) = 2$, $w(123) = w(124) = w(134) = w(234) = w(13) = w(14) = w(23) = w(24) = 1$, and $w(S) = 0$ otherwise. Clearly, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is coalition-rational, and so (N, w) is a balanced game. Let w^1, w^2, w^3, and w^4 be the same as w except that $w^1(134) = 2$, $w^2(234) = 2$, $w^3(123) = 2$, and $w^4(124) = 2$. Suppose x is a coalition-rational allocation for (N, w^1). Then $0 = w^1(2) \leq x_2 = w^1(N) - x_1 - x_3 - x_4 \leq w^1(N) - w^1(134) = 0$, and so $x_2 = 0$. Furthermore, $x_3 = x_2 + x_3 \geq w^1(23) = 1$ and $x_4 = x_2 + x_4 \geq w^1(24) = 1$. It now follows from efficiency and $x_1 \geq 0$ that $x_3 = x_4 = 1$. Hence, $(0,0,1,1)$ is the only possible coalition-rational allocation, and it is easily verified that $(0,0,1,1)$ is coalition-rational. Since α is coalition-rational, $\alpha(w^1) = (0,0,1,1)$. Analogous arguments imply $\alpha(w^2) = (0,0,1,1)$, $\alpha(w^3) = (1,1,0,0)$, and $\alpha(w^4) = (1,1,0,0)$.

Notice that $w(134) < w^1(134)$ and $w(S) = w^1(S)$ for all $S \neq 134$. Since α is coalition-monotone, $\alpha_1(w) \leq \alpha_1(w^1) = 0$. Analogous arguments imply $\alpha_i(w) \leq \alpha_i(w^i) = 0$ for $i = 1,2,3,4$. But this violates the efficiency of $\alpha(w)$. This contradiction proves the theorem. □

Theorem 6.1 shows that there are limits to the number and strength of the fairness properties we can impose. If players are able to choose whether or not they will cooperate in a joint economic venture, the use of a coalition-rational method, such as the weighted nucleoli, is indicated, and if changes in coalition worths are unlikely, then the coalition-monotone property is not crucial. As argued by Felsenthal and Machover [11], the measurement of voting power should use a coalition-monotone method such as the Shapley value and other, as of yet unidentified, weighted contribution values; the loss of the coalition-rational property is bolstered by our earlier remark that coalition-rational methods do not recognize the power differences among the dictator and veto power games.

Theorem 6.1 also suggests that there may be greater possibilities if we can restrict ourselves to smaller classes of games. For example, we have already noted that the Shapley value is coalition-rational and coalition-monotone on convex games. Housman and Clark [15] showed that there are many coalition-rational and coalition-monotone allocation methods (the nucleolus and per capita nucleolus being two of them) when restricted to three-player balanced games.

Some collections of properties characterize a single allocation method. Although we want methods to satisfy properties on large classes of games, it is useful to characterize methods on the smallest class of games possible.

player-separable requires that the allocation of a sum of two games, where one game is additive, is the sum of the allocations for the two games. Mathematically, it would perhaps be elegant to strengthen this property by removing the restriction that one game be additive. This has a real-world fairness interpretation when a coalition game could be considered the sum of separate games (e.g., the savings game may be the sum of savings from land acquisition, materials purchasing, and labor). The allocation from the sum game and the sum of the allocations from the separate summand games should be the same. Otherwise, the method of accounting has an effect on the allocation.

DEFINITION 6.2 (Additive Property). An allocation method α is *additive* on a set Γ of coalition games if $\alpha(N, v+w) = \alpha(N,v) + \alpha(N,w)$ for all $(N,v) \in \Gamma$, $(N,w) \in \Gamma$, and $(N, v+w) \in \Gamma$.

The egalitarian method and Shapley value are additive. No other a-synergy value is additive. The a-nucleoli are piecewise additive, that is, an a-nucleolus is additive on each element of some partition (the partitioning depending on a) of the space of all games into convex sets. The following theorem was first proved by Shapley [29].

THEOREM 6.3. *If an allocation method α is unbiased, player-separable, and additive on convex games, then α is the Shapley value.*

PROOF. The Shapley value clearly satisfies the four properties. We showed in section 1 that given any game (N,w), we can write $w = \sum_{T \subseteq N} d_T u^T$, where u^T is the unanimity game on T and d_T is a number dependent on w. Since α is unbiased and player-separable, $\alpha(N, d_T u^T) = (d_T/|T|)\chi^T$. Since α is additive, $\alpha(N,w) = \sum_{T \subseteq N} \alpha(N, d_T u^T)$. This shows that the method is uniquely determined at each game. We can limit our characterizing class to convex games because the unanimity games are convex and any convex game can be written as a linear combination of unanimity games. □

We can strengthen aggregate and coalition monotonicity one step further.

DEFINITION 6.4 (Strongly Monotone Property). An allocation method α is *strongly monotone* on a set Γ of coalition games if $\alpha_i(N,v) \leq \alpha_i(N,w)$ for all $(N,v) \in \Gamma$, $(N,w) \in \Gamma$, and $i \in N$ satisfying $v(S) - v(S\backslash\{i\}) \leq w(S) - w(S\backslash\{i\})$ for all $S \subseteq N$ containing i.

Young **[35]** proved the following theorem by inducting on the number of non-zero terms in the sum $w = \sum_{T \subseteq N} d_T u^T$.

THEOREM 6.5. *If an allocation method α is unbiased and strongly monotone on superadditive games, then α is the Shapley value.*

We have suggested what a fair allocation method should select when games are added (player-separable and additive) and when some of the coalition worths are changed (aggregate, coalition, and strongly monotone). The last property suggests what a fair allocation method should select if some players want to renegotiate their payoffs amongst themselves. Suppose an allocation x has been proposed, the players in a coalition T wish to renegotiate amongst themselves, and the remaining players are satisfied with their payoffs. To determine its worth in the renegotiation, a coalition S of T should be able to join with a coalition Q outside of T as long as the players in Q are compensated in accordance with x. Presumably, S will choose Q to maximize its worth.

DEFINITION 6.6 (Reduced Game). Let (N, w) be a game, T a coalition, and x an allocation. The *reduced game* with respect to T and x is the game $(T, w^{T,x})$ defined by

$$w^{T,x}(S) = \begin{cases} 0 & \text{if } S = \varnothing \\ \sum_{i \in T} x_i & \text{if } S = T \\ \max_{Q \subseteq N \setminus T} \left(w(S \cup Q) - \sum_{i \in Q} x_i \right) & \text{otherwise} \end{cases}$$

For the savings game (N, w), the coalition $T = \{A, B\}$, and the egalitarian allocation $x = (35, 35, 35, 35)$, the reduced game $(T, w^{T,x})$ is defined by $w^{T,x}(AB) = x_A + x_B = 70$, $w^{T,x}(A) = \max\{0, 0-35, 0-35, 84-70\} = 14$, and $w^{T,x}(B) = \max\{0, 0-35, 0-35, 0-70\} = 0$. Notice that $\xi_A(T, w^{T,x}) = 42 \neq 35 = \xi_A(N, w)$. If the egalitarian method is used to allocate, renegotiations among smaller coalitions may result in inconsistencies. Formally, the egalitarian method is not reduced game consistent.

DEFINITION 6.7 (Reduced Game Consistent Property). An allocation method α is *reduced game consistent* on a set Γ of coalition games if $(N, w) \in \Gamma$, $T \subseteq N$, and $T \neq \varnothing$ implies $(T, w^{T,x}) \in \Gamma$ and $\alpha_i(T, w^{T,x}) = \alpha_i(N, w)$ for all $i \in T$.

Notice that the reduced game consistent property insists that the reduced games created are in the focal class of games. This may force us to expand the class of games under consideration. For example, the 3-player veto power game is superadditive and simple, but the reduced game with respect to any pair of players and the egalitarian allocation is neither superadditive nor simple. So, the egalitarian method is not reduced game consistent on superadditive or simple games.

Suppose α is a player-separable, strongly unbiased, and reduced game consistent allocation method. Suppose $x = \alpha(N, w)$ for the savings game (N, w). Since x is strongly unbiased, $84 - x_C - x_D > 0$, and so $w^{AB,x}(AB) = x_A + x_B$, $w^{AB,x}(A) = 84 - x_C - x_D$, and $w^{AB,x}(B) = 0$. By Theorem 2.7,

$$\begin{aligned} \alpha_B(AB, w^{AB,x}) &= \xi_B(AB, w^{AB,x}) \\ &= \frac{1}{2}(x_A + x_B - (84 - x_C - x_D)) \\ &= \frac{1}{2}(140 - 84) = 28. \end{aligned}$$

Since α is reduced game consistent, $x_B = 28$. Similar arguments using reduced games on AC and AD require that $x_C = 22$ and $x_D = 16$. By the efficiency condition, $x = (74, 28, 22, 16)$, which is the (pre)nucleolus.

THEOREM 6.8. *If an allocation method α is anonymous, player-separable, proportionate, and reduced game consistent on all games, then α is the prenucleolus.*

Sobolev [**31**] proved this theorem. Unlike the motivating savings game example, the proof requires the focal game to be embedded in multiple ways inside an enormously larger game. Orshan [**24**] was able to obtain the same conclusion using unbiased instead of anonymous.

7. Summary

We have described several classes of coalition games, allocation methods, and fairness properties. We have determined which methods satisfy which properties on which classes of coalition games. We have shown that a set of fairness properties can be mutually inconsistent on a class of coalition games. We have also shown that some sets of fairness properties can uniquely characterize an allocation method. No method appears to be perfectly fair for all circumstances, and so it is important to identify the fairness properties most appropriate for any given class of models. For example, the coalition-monotone property is particularly compelling in measuring voting power while the coalition-rational property is particularly compelling when economic agents can choose whether or not to cooperate. A careful examination of particular situations are likely to lead to different, and perhaps new, fairness properties.

This leaves many questions unanswered or even as yet unasked. There are other classes of games, other allocation methods and set-valued solution concepts, and other fairness properties either described in the literature (see the references cited in the introduction) or yet to be discovered. In addition to coalition games, there are other mathematical models of cooperation such as nontransferable utility games [**3**, chapter 55], partition function form games [**5**], and partially defined games [**14**].

For the reader who would like to make a contribution, choose a situation and a corresponding model. Examples would be airport landing fees modeled as convex coalition games or Presidential power in legislation modeled as a simple coalition game. Intuit what seems correct. Perhaps it is that each airplane landing should pay an equal share of the cost for a runway only as long as was needed or that the President should have at least as much power as all of the Senators combined. Apply known allocation methods such as the Shapley value and nucleolus. Think more deeply about the properties that an allocation method should have in your context and characterize which allocation methods satisfy the properties you have defined. Perhaps you will characterize a method that confirms your intuition or find that your proof contradicts your intuition. Either way, you will have learned something and made a contribution. So, ask your own questions and find the answers!

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Department of Mathematics, Goshen College

E-mail address: dhousman@goshen.edu

URL: http://www.goshen.edu/~dhousman