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FACILITATING LEARNING EVENTS THROUGH EXAMPLE GENERATION[1]

ABSTRACT. This study deals with the initial understanding that advanced undergraduate mathematics students exhibit when presented with a new concept in an environment requiring self-generation and self-validation of instances of the concept. Data were collected in spring of 1995 through interviews with 11 third and fourth year undergraduate mathematics students. We discuss the data from the perspective of the student's concept image and introduce the notion of learning event to indicate when a student communicates and applies a new understanding of a concept. We infer that the students in our study who employed an example generation learning strategy were more effective in attaining an initial understanding of the new concept than those who primarily employed other learning strategies such as definition reformulation or memorization.

1. INTRODUCTION

A central goal of upper level undergraduate mathematics courses is to enable students to understand and use mathematical abstraction and formalism. For example, in the typical real analysis or algebraic structures course, students are expected to absorb scores of formal definitions and theorem statements which are then used to prove or disprove many other statements given in homework and exam problems. We are interested in how students develop an initial understanding of a formal concept. Our focus on the initial stages of learning new concepts is important because of the large number of such concepts we expect students to learn during a typical upper level undergraduate mathematics course and because a student's initial understanding of a concept often persists despite the presence of examples and information which conflict with this initial understanding (Davis and Vinner, 1986).

We adhere to the constructivist perspective that students actively construct personal interpretations of knowledge (Ernest, 1991; von Glasersfeld, 1984), and use the theory of concept image/concept definition originally proposed by Vinner and Hershkowitz (1980), developed by Tall and Vinner (1981), and modified by Moore (1994). A *concept definition* is "a formal verbal definition that accurately explains the concept in a non-circular way, as might be found in a mathematics textbook" (Moore, 1994:

structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall and Vinner, 1981: p. 152). That portion of the concept image which is activated at a particular time is called the *evoked concept image*. Note that we consider verbal reformulations of a concept by the student as part of the student's evoked concept image, what Tall and Vinner refer to as the personal concept definition. Tall and Vinner use their theory to examine the "potential conflict factors" which arise from secondary school instruction related to sequence limit, function limit, and function continuity. Potential conflict factors include insufficiently developed, incorrect, and contradictory parts of the concept image.

We are interested in the initial development of a student's concept image immediately after being presented with a concept definition. Because of the personal and internal nature of a concept image, a description of its state and development must be inferred by the student's verbal and written communications. Instead of well-formed concept images, our object of study are significant changes in the evoked concept image. In this context, interviewer questions, comments, and body language could have a profound effect upon the concept image formed by the student. A student explanation of his or her understanding of the concept definition can give a glimpse of the student's concept image, but it may also cause further development in the student's concept image. Given the impossibility of directly observing the developing concept image, we will say that a *learning event* has occurred when the student communicates and applies a new understanding of the concept. A communication could be verbal or written and could be an example, a reformulation, or an explanation. An application involves a use of the new understanding to solve a new problem or re-explain the answer to a previous question. We suggest that such observable learning events correspond to the unobservable changes in concept image.

In addition to the concept definition – concept image scheme outlined above, Moore (1994: p. 252) discovered a third aspect of concept understanding: *concept usage*, "which refers to the ways one operates with a concept in generating or using examples or in doing proofs". These three aspects – concept definition, concept image, and concept usage – constitute what Moore calls the concept-understanding scheme. We have also found this last aspect, concept usage, to be important in learning and working with a new concept. Our data reveals that example usage, particularly example generation and verification, is crucial for understanding a new concept.

Moore's work (1994) with undergraduate students in the U.S. dealt with the transition from calculus to formal proof such as that encountered in courses in linear algebra, abstract algebra, and analysis. Some of the sources of difficulty students had in doing proofs were inability to state the definitions, little intuitive understanding of the concepts involved, inadequate concept images, and inability or unwillingness to generate and use their own examples (Moore, 1994: p. 251). Our study primarily deals with this beginning phase during which students acquire knowledge of new concepts from definitions before applying them in proofs. As pointed out by Moore, students must be able to use definitions to obtain the overall structure of proofs. In Selden and Selden (1995), it is argued that students inability to "unpack" informally written mathematical statements into the language of predicate calculus precludes their ability to discern the top-level logical structure of the proof ("proof framework") of a theorem and thereby determine its correctness. From the perspective of a developmental progression, we see the acquisition of concepts from definitions and examples as preceding the determination of proof frameworks, which in turn is followed by the construction of proofs.

We attempted to study the evoked concept images arising when a student is learning a new mathematical concept and what factors facilitate movement towards a correct understanding of the concept. Students were presented with a formal written definition (concept definition) and observed reacting to various stimuli including neutral requests to explain the student's understanding, verification questions, and interviewer leading questions and explanations when necessary. We attempted to identify learning events, explain the stimuli prompting learning events, and determine the effects of long-established concept images.

2. THE CURRENT STUDY

Two interviews were conducted at Allegheny College by the authors: a preliminary interview constituting a pilot study in the fall of 1994 and a second interview during the spring of 1995. The authors did not discuss the results of the pilot interview with the participants and the present study is based on the results of the second interview. We interviewed 11 students (6 male, 5 female) and based upon their past performance in mathematics classes, the 11 students constituted four ability level groups in descending order as follows (names have been changed): Ann, Andy, Art; Betty, Brenda, Beth; Carl, Carol, Chad; and Dan, Don. All but one of the students were mathematics majors[2] who had successfully completed a foundations course similar to the one in Moore's study, linear algebra, a seminar on set theory and the foundations of analysis, and introductory who had taken linear algebra and an automata course. Brenda was a junior while the rest were seniors.

Allegheny College is a small (student population size is under 2000) liberal arts institution which emphasizes close interaction between students and faculty (the student to faculty ratio is 11 to 1). The students in our study had been exposed to a variety of teaching styles including the lecture method, lecture supplemented by computer laboratory sessions using *Mathematica*, lecture with some small group activity, and cooperative learning with small groups. These were students who were approaching the end of their undergraduate studies and were intending to continue their mathematics education in graduate studies (1–2 of the students) or to be employed in ways that use mathematics such as teaching secondary school or actuarial work (9–10 of the students).

The students were interviewed individually with the sessions lasting from 20 minutes to an hour and consisted of observing and interacting with each student as he/she learned a new mathematical concept. The interview experience was independent of any course the student was currently taking and the concept definition was not one to be found in any textbook used in any of the courses the students had previously taken. Certainly the students had had previous experience with the ideas used to define this new concept, but the purpose of the interview was to observe how students learn a new concept. The format for the interview consisted in presenting the student with a concept definition on a sheet of paper (definition page) with plenty of space for the student to reformulate or generate examples of the definition. This was followed by presenting the student with several more pages containing questions which asked the student to perform various tasks.

The students were encouraged during the interview to write on the pages and to "think out loud". We often asked students to explain their thinking whether or not previous work appeared correct to us. This allowed us to observe aspects of their evoked concept images, and helped to prevent students from inferring that only incorrect arguments on their part would be challenged. They could refer to any previously given pages throughout the interview. The interviews were audiotaped and detailed notes were kept by the interviewers. During the interviews, the authors tried not to intervene except in situations where it became clear that no further progress would be made. The first level of intervention were requests for clarification of student comments and written work. In some cases, the interviewer had to remind the student of the meaning of some of the terms used in the concept definition (we will refer to these as *base concepts*). This was done so these students would have an opportunity to progress to the later questions and finish the interview.

The five interview pages are reproduced below. The *definition page* contained the following instructions with the concept definition printed at the top. In addition, the instructions, but not the definition, were read to the student by the interviewer.

Instructions. You will have a few minutes to study the following mathematical definition. You may write on this paper, and you will always have access to this paper during the interview. It would be helpful if you would state or write what you are thinking as much as possible.

Definition. A function is called *fine* if it has a root (zero) at each integer.

The concept definition was purposely designed to allow for a variety of evoked concept images of the base concepts; function and root. Functions may be thought of symbolically, graphically, numerically, and verbally (Leinhardt, Zaslavsky and Stein, 1990), and as objects or processes (Breidenbach, Dubinsky, Hawks and Nichols, 1992). Since the function domain was left unspecified, students were implicitly given the opportunity to choose a domain: complex, real, or some subset of these sets. Roots can be thought of as inputs evaluating to zero, the end result of a factorization procedure, or as intersections of the function's graph with the x-axis. All of the above described evoked concept images (and more) were observed in the students.

After five to ten minutes the students were then presented with the *generation page*. Based upon the results of our pilot study we determined that student generated examples and reformulation played important roles in eliciting learning events. On the generation page we attempted to induce example generation and concept reformulation in those students who had not already employed these learning strategies.

Instructions. Please answer the following questions.

1. Give an example of a fine function and explain why it is a fine function.
2. Give an example of a function which is *not* fine and explain why it is not fine.
3. In your own words and/or pictures, explain what a fine function is.

The next page was a *verification page* which provided functions for the student to determine if they satisfied the definition of fine function.

Instructions. Determine with explanation, which of the following functions are fine.

1. $f(x) = \sin(\pi x)$
2. $f(x) = x^2 - x$
3. $f(x) = 0$
4. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$
5. $f(x) = \tan\left(\frac{\pi}{2}x\right)$

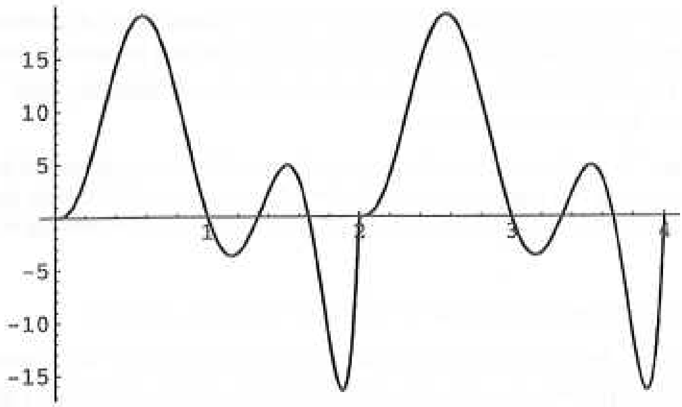

Figure 1.

The *conjecture page* required the students to determine the validity of the following four statements about fine functions, each of which is false. We were interested in the extent to which students engaged in concept usage by either finding a proof or exhibiting a counterexample.

Instructions. Determine, with explanation, which of the following conjectures are true.

1. No polynomial is a fine function.
2. All trigonometric functions are fine.
3. All fine functions are periodic.
4. The product of a fine function and any other function is a fine function.

Finally, the last page consisted of a single question to survey student perceptions of the interview and how it compared with their usual approach to learning a new concept.

Instructions. Please answer the following question.

How did this learning experience compare with your approach to learning new concepts in upper-level, undergraduate mathematics courses?

3. RESULTS

3.1. *Initial Strategies and Learning Events*

We found four basic learning strategies being used by students when presented with the concept definition: example generation, reformulation, decomposition and synthesis, and memorization. In general, the initial sophistication of the evoked concept image of fine function was highest among the example generators and decreased with each successive strategy

Ann, Andy, Chad, and Dan all attempted to generate examples that illustrate the concept. Ann and Andy exhibited similar learning events starting with a graph of the constant zero function and moving to a sinusoidal graph with integer x-intercepts. Their evoked concept image of fine function is the most well-developed consisting of graphs of continuous periodic functions with integer x-intercepts. Dan started to exhibit a similar set of learning events by starting with a graph of the constant zero function and a Cartesian coordinate system with integer x-intercepts shaded, but he then states that a fine function must be of infinite degree. The graphs are not recognized as examples of fine functions but only as a technique of possibly generating an example necessarily expressed as a formula. Chad's initial evoked concept image of a fine function was a nonconstant polynomial with integer coefficients. He interpreted the phrase "root(zero) at each integer" to mean that each integer coefficient in the polynomial must have an x associated with it. For example, he worked with expressions like $3x - 3x$ which satisfied his interpretation of root, but didn't realize that this was an example of a fine function and hence this was not a learning event for him.

Art and Beth obtained a (symbolic) reformulation: $f(n) = 0 \ \forall n \in Z$ and $f(-1) = 0$, $f(0) = 0$, $f(1) = 0$, $f(2) = 0, \ldots$, respectively. Brenda began with a (verbal) reformulation in terms of factoring and then attempted to generate examples. These students obtained learning events limited to algorithmic procedures for checking whether a given function was fine and did not spent much time on the definition page.

Betty and Carl, by underlining parts of the definition and asking about the meaning of root, showed evidence of using decomposition and synthesis: the process of "breaking down" a concept into its constituent parts, examining the meaning of each of these parts, and finally recombining. An example supplied by the interviewer prompted Betty to give a verbal reformulation.

Carol and Don simply read the definition but did not write on the definition page. They remarked that they often use rote memorization in learning a concept. Studies have shown that students perceive memorization as a major component of mathematics learning (Spangler, 1992). These students did not exhibit learning events exclusively stimulated by the concept definition. They did not complete the generation page without interviewer intervention, and were the ones who most often misinterpreted the meaning of the definition. They found the interview to be structured differently from their normal learning process which included an initial examination of externally provided examples and/or explanations.

3.2. *Effect of Suggesting Learning Strategies*

The generation page asked students to employ example generation and reformulation learning strategies. These questions did not prompt learning events in students with an already well developed understanding (e.g. Ann) nor in students who required explanation from others (e.g. Don). The questions on the generation page appear to be most effective when suggesting a *change* from one of the example generation or reformulation strategies to the other. Art, Brenda and Andy provide illustrations.

Art began as a reformulator and remarked that his symbolic reformulation was the first idea that occurred to him. When asked for an example (#1 on the generation page), he had his most significant learning event: graphing what appeared to be $y = \sin(\pi x)$ and shortly afterward giving a verbal description of a graphical example which was continuous and non-periodic ("Because anything that goes up some place on any side, I mean the amplitude can be quite extreme, but it has to come back to the x-axis").

Brenda was an example generator for much of the interview. She had offered $f(x) = x - x$ as an example of a fine function for #1 on the generation page but was unsure and crossed out her expression. After going through a socratic question and answer period with the interviewer, she agreed with the interviewer that her example was fine, but she immediately changed her focus writing down the expression $f(x) = (x - z_1)(x - z_2)(x - z_3)$ $z_1, z_2, z_3 \in Z$ and describing the concept of fine function in terms of an "infinite factorization" of the function where the factors are of the form $x - z$ for all integers z. However, Brenda was not sure about the concept of infinite products, so she worked with finite products attempting, in essence, to "factor" her original example $x - x$. After a period of trial and error, she gave up her effort at factoring the zero polynomial. At this point, Brenda's understanding was adequate to give $f(x) = x^2 - 4$ as an example of a functions that was not fine.

Brenda obtained her most significant learning events when forced to reformulate (#3 on the generation page). In response to the interviewer asking for a picture, Brenda drew an xy-coordinate system with the x-axis shaded: "What I'm thinking, I'm not sure if it's right, but if it has a root at every integer, then it would just be like the x-axis when graphed. Am I thinking right?" Brenda was still uncertain and stated several verbal reformulations of the concept definition partially prompted by interviewer questions. She finally states, "So the zero function would work because it obviously has a root at each integer, and you're disregarding whatever happens at any other real number besides the integers". Although Brenda an example of a fine function, she was not convinced until after making these pictorial and verbal reformulations. A short while later she exhibited another learning event:

> Or it could be like the x-axis or what comes into my mind is something like the Dirichlet function, maybe real erratic with, with the [...] integers, say negative four, negative three, negative two, negative one, zero, one, two, three, four, five, six, all other roots it doesn't matter [...] if it was the sine function or just all up or down as long as it was a function. We are not really concerned what happens here (pointing to the spaces in between the integer marks on the x-axis).

Brenda then drew a sinusoidal curve connecting the integer intercepts.

An example generator on the definition page, Andy first answered #3 on the generation page: "A function is fine if it crosses the x-axis at every integer". He had previously drawn a sinusoidal function on the definition page, but now he drew coordinate axes with dots on the horizontal axis integers. He appeared to have a revelation, added a new (discontinuous) fine function example ($f(x) = 0$ if x is an integer and $f(x) = 1$ otherwise) and graphed the example.

For Art, Brenda, and Andy, the questions on the generation page suggested *changes* from one strategy to the other resulting in a shift towards a graphical representation of function which then facilitated the generation of examples and consequently of learning events. For some students, the questions on the generation page had little effect in moving them towards a visual approach to finding an example. Dan's example of a fine function is the expression $f(x) = 0 \cdot x$ with the explanation "because for any integer x $f(x) = 0$". He also gave a correct counterexample (quadratic polynomial) and answered #3 by writing "A fine function is a function $f(x)$ such that, any integer x yields the result $f(x) = 0$". On the other hand, Beth spent a long time in silence considering question #1 on the generation page. When asked what she had been thinking, Beth replied "I'm thinking that it [an example] has to be some function where when you plug in any integer, the function is going to equal zero". When the interviewer suggested that she skip to #2 and #3, Beth provided $f(x) = x$ as an example of a function that is not fine and gave a reformulation in writing similar to her reply above. After some more thought, she exhibited a learning event by writing for question 1: "$f(x) = 0$ for any integer, the function will always equal zero". She was not confident in her answers and remarked "I don't know if I have the definition right at the beginning".

What did students do when they couldn't find an example? In some cases, students changed the meaning of a concept. Both Carl and Betty's initial evoked concept image of function was that of polynomial functions which they realized have only a finite number of roots. Unable to obtain mean that a fine function must vanish at each integer in its domain which may not include all integers. However, after receiving clarification from the interviewer of root and zero and reflecting on the definition, they realized that this interpretation was not correct, and eventually shifted from a formula interpretation to a graphical view of function at which point they exhibited learning events by providing the x-axis (zero function) as an example of a fine function.

Carol displayed how long the changed meaning can last when there is no intervention to stop the development of incorrect concept images. Carol initially wrote $f(x) = x^2 + 2x + 1$ as an example of a fine function. After factoring this expression, she used it as a non-example because "This only has a root at one integer, -1". She then wrote $y = x$ as an example because "This has a root at each integer x". On question 3 of the generation page she graphed $y = x$ and wrote "I think that this means a function for which there will always be a root, no matter which integer we focus on". Carol's concept of root seemed to be changing from the notion of factoring to something else. The something else became clearer on the verification page where she had to determine whether functions are fine or not. She determined that functions 1, 2 and 6 on the verification page were fine because they were "continuous everywhere" and so had "a root at each integer". Functions 4 and 5 were not fine because "it wasn't continuous everywhere" and 'it's only defined at here and here" (pointing at the continuous sections of a graph of the tangent function), respectively. Her answer that function 3 was not fine may at first seem to be inconsistent, but Carol explained, "When I thought zero, ... I just thought of a point", and so $f(x) = 0$ was viewed as a point which did not match her image of a continuous function.

3.3. *Effect of Evoked Base Concept Images*

We have seen that a variety of concept images are evoked for the function and root base concepts, and incomplete or incorrect concept images have significant effects upon student understanding of the fine concept. We have already described how Carol's identification of "root" with "continuity" resulted in her long misunderstanding of fine functions. An incorrect evoked image shared by three students involved thinking of the graph of the zero function as a point rather than a horizontal line. Because of this incorrect image, Carol and Don had difficulty understanding that the zero function is fine, and Beth believed that the zero function is not periodic.

Most students' initially evoked concept image of a function was a (non-constant) polynomial. This is consistent with findings that even teachers of mathematics have a tendency to think only in terms of continuous consider discontinuous functions (Hitt, 1994). In general, students have a strong tendency to think algebraically rather than visually even in situations where they are pushed towards visual processing (Eisenberg and Dreyfus: p. 29). Without an expansion in this limited evoked concept image of a function, it is impossible to find an example of a fine function. One expansion occurred with the transcription of notation. Note how Brenda early on writes down $f(x) = x - x$. Carl does the same and Dan writes $f(x) = 0 \cdot x$. It takes them a while to realize that this notation represents the constant function 0. Another potential expansion is the idea of an infinite product of factors expressed by Dan and Brenda, but this is related to their evoked concept image of function as polynomial function and does not result in any viable examples.

The most significant expansion in the evoked concept image of function, in terms of being associated with learning events, is the use of visualization in the sense of Zimmermann and Cunningham (1991: p. 3): "Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding". No student was able to obtain an example of a fine function other than the zero function until they evoked a graphical image of the function concept. Ann and Andy immediately obtained graphical images of functions during the definition page. Art, Brenda, Betty, and Carol were prompted to use a graphical image by #3 on the generation page. The remaining students did not use a graphical image of function until the verification page. These students did not generate examples of fine functions and were reluctant to engage in example generation or reflection later on when working on the conjecture page.

3.4. *Role of Examples*

The generation of and reflection on examples provided powerful stimuli for eliciting learning events. We illustrate with Betty, Ann, Dan, and Don.

Betty, whose initial strategy had been reformulation, had great difficulty obtaining an example on the generation page. She eventually produced the zero function and graphed it on a coordinate axis. After some reflection, she exhibited another learning event:

> So it [the graph of a fine function] probably could even be a graph with holes in it. It doesn't matter what the rest of the graph looks like in between the integers as long as at the integers, it's equal to zero.

Betty remarked that she didn't know how to produce a formula to describe such a function, but when prompted for a picture, she drew a coordinate

"[...] it could even be a sine function in between as long as at every integer it would equal zero". On the conjecture page, she connected with this learning event and provided another by producing the product of x and the Dirichlet function as a counterexample to conjecture 3. Only students who had obtained graphical examples of fine functions were able to obtain a counterexample to conjecture 3 even though symbolic counterexamples are easily given (e.g., $f(x) = x \sin(\pi x)$).

In Section 3.1, we described Ann's rapid succession of learning events and well-developed evoked concept image developed through example generation. Ann moved rapidly and correctly through the generation and verification pages. She provided an abstract pictorial reformulation of the fine function concept that shows an awareness of domain considerations, but she stated that she was "uneasy" because "I'm not real sure that I understand this coordinate picture (referring to a drawn set of coordinate axes)". On the conjecture page, Ann exhibited several learning events. She quickly gave the zero function as a counterexample to conjecture 1. When asked by the interviewer whether the modified conjecture *No non-zero polynomial function is a fine function* is true or false, Ann answered correctly with a correct argument based on the degree of the polynomial. She correctly showed that conjecture 2 was false by generating her own counterexample: $\sin(x)$. In considering conjecture 3, Ann stated that at the beginning of the interview, she believed that this statement was true, but now she is uncertain. After glancing back at the verification page, she offered the Dirichlet-type function (#4) as a counterexample. When asked to explain the non-periodicity of this function, she quickly thought through the example and convinced herself that this function actually was periodic.

Ann: Yea, it's true. I guess that's where my confusion comes in because that's one of the first things I thought (on the definition page), and I couldn't think of one (fine function) that wouldn't be periodic. Perhaps there is one, and that's why I'm not sure.

I: How would you try to come up with an example that wasn't periodic?

Ann: I guess that's when I was starting to think about discontinuous functions ... where I could just say f of x equals zero if x is an integer, and f of x equals whatever. Let's see, yeah, equals x for x a real number that is not an integer (starts drawing axes and the graph of the defined function) that would do it because you would have your zeros here, here, and here. And then you would basically have this line with breaks. It is then periodically zero and then you have this thing happening (pointing to the line).

I: All fine functions are periodic. Are you saying that this is true or false?

Ann: All continuous fine functions are probably periodic. Probably (chuckle).

I: Oh, okay. You are saying that if it is discontinuous, then..

I: What is it about the continuity that is making you think that the statement would then be true?

Ann: Well, I guess not. Looking at the . . . I could just as well keep having the amplitudes in different directions or having them this way, doubled this way.

I: Okay.

Ann: It's good to have an expression, but. So maybe not.

I: So maybe not. Are you comfortable with that answer now? You were just examining the conjecture "all continuous fine functions are periodic", and you are thinking that is false now, or is that true?

Ann: (with an expression of resignation) Yeah, I'm thinking that's false now.

By first considering a discontinuous periodic fine function (Dirichlet), then varying it (essentially multiplying by x) to produce a discontinuous non-periodic fine function, Ann saw that a continuous non-periodic fine function may exist (she described something like $x\sin(\pi x)$). However, this required some effort on her part to overcome the dissonance between her initial evoked concept image of fine function as a periodic function and her visualization of a counterexample. She was also uneasy about her example because she did not have a formula for it. This sequence of interchanges illustrates several of Ann's learning events and example usage in generating and modifying examples to ultimately produce a counterexample to the statement. More importantly, it shows how example generation allowed her to change her concept image of fine function.

In some cases, students made little progress until they were presented with the candidate functions on the verification page. These functions appear to have evoked parts of the students' concept image of function that had previously been untapped. Upon seeing the sine function example (#1), Dan stated "I'm seeing the sinusoidal graph, or how I remember it, in my mind". He required a calculator to check his impressions but then stated, "I see a broader range of fine function than I used to" and remarked that the zero function was the only example he could think of prior to seeing the verification page. On the verification page, Don, who had previously not been able to produce a single example of a fine function, was able to quickly and correctly determine which of the candidate functions were fine. However, on the conjecture page he only answered #1 and #2 with conviction, using counterexamples from the verification page. With the other conjectures he was not able to make much use of the examples and was even reluctant to consider examples when suggested by the interviewer.

Students who consistently employed example generation (Ann, Andy, Betty and Brenda) had more learning events, were able to encapsulate more examples into their concept image of fine function, and were more able to use these examples than those who primarily used other learning strategies.

the definition page during the interview, and Beth was reluctant to consider examples after giving a faulty argument for conjecture 4. On the generation page, Art switched to an example generation mode, exhibited a learning event by graphing $y = \sin(\pi x)$ and even described a graphical example which was continuous and non-periodic (similar to $x \sin(\pi x)$). However, Art did not consider this example at all when attempting to find a counterexample to conjecture 3, and instead referred to the Dirichlet function (#4) provided on the verification page (which is not a counterexample).

3.5. *Persistence of Initial Learning Events*

In some of the interviews, we observed that students' initial learning events dominated how they viewed the fine function concept. An illustration of this persistence is Ann's initial evoked concept image of fine function as a periodic function which caused her some conflict later when considering whether a fine function must be periodic (see Section 3.4). Another illustration was Beth's resistance to inspecting examples after deciding that conjecture 4 was true and giving a proof based on her initial reformulation of the definition in terms of function evaluation. Carol's evoked concept image of fine function as a continuous function was very persistent, lasting from the definition page through the verification pages, and required much intervention by the interviewer to effect a new learning event.

Chad's initial evoked concept image of fine function as the zero function causes him some conflict when considering #6 on the verification page. In spite of the extensive interviewer prompting given in response to his difficulty in understanding the sine and Dirichlet functions, and although he eventually "agrees" after some interviewer prompting that the graphical presentation for #6 is fine, these experiences make little impression on him. When working on the conjecture page, he was stymied when he could not produce an example of a fine function other than the zero function and hence, even though he answered yes for #6, no learning event occurred since he was not able to use this example on later questions. In fact, even at the end of the interview Chad was still uncertain of his understanding of fine function and remarked "There aren't too many fine functions in this world. Just being able to come up with one off the top of your head is very difficult".

During the interviews we were struck by the relative ineffectualness of instructor intervention. Some learning events did occur after interviewer prompting, but we were surprised to find how little an effect this activity had in stimulating student learning events. For example, Brenda offered $f(x) = x - x$ as an example of a fine function but was unsure and answer period with the interviewer, she agreed that her example was fine, but immediately changed her focus. The interviewer led Brenda to an agreement that the zero function was a fine function yet Brenda immediately moved onto other ideas, and it was not until much later, through self-discovery, that Brenda realized that the zero function really was an example of a fine function. In the case of Don, extensive prompting produced little understanding although it did keep him focused on the task and perhaps led to understanding later when he encountered the verification page. Of course, Carol would have never corrected her erroneous notion of fine meaning continuous during the interview without intervention by the interviewer. In summary, interviewer prompting was most effective when students were completely lost or when students were having difficulty articulating a learning event.

4. CONCLUSIONS

When do learning events occur? Based upon our interviews, we found that student concept usage through generating, verifying and reflecting on examples either stimulated or were associated with learning events. Students who initially employed the example generation strategy learned a significant amount having only been given the concept definition. In general, those students who engaged in example generation and reflection during the interview were able to attain a more complete understanding of the fine function concept. These students were able to incorporate a greater variety of functions (discontinuous, periodic continuous, non-periodic continuous, etc.) into their concept image of fine function and were able to use these examples in their explanations.

Students who primarily employed memorization or decomposition and synthesis as their learning strategy most often misinterpreted the definition and either did not complete the conjecture page or gave answers (usually guesses) but with little or no explanation to support their answers. Students who primarily employed reformulation as their learning strategy developed algorithms to quickly verify whether candidate function were fine; however, they had difficulty identifying counterexamples to conjectures even when the counterexample existed on the verification page. Students who primarily employed example generation were best able to identify the correctness of conjectures and provide explanations. It may be that reformulation plays a more important role in the construction of proofs (see Moore (1994) and Selden and Selden (1995)).

Our findings suggest that it may be beneficial to introduce students to them verify and work with instances of a concept before providing them with examples and commentary. The students in our study who were able to move freely between reformulation and example generation were most likely to be successful in understanding the fine function concept. Some students were reluctant or unwilling to engage in example generation, or more generally, example usage. These students were unsure of their answers and repeatedly sought confirmation from the interviewers. Perhaps engaging students in example usage activities while introducing students to new concepts can promote this strategy and encourage students to consider more carefully the meaning of definitions. However, questions given to students need to be thought through carefully because of the persistence of the student's sometimes erroneous initial concept image and the non-persistence of instructor intervention.

Finally, we view the structure of the interview as a useful methodological approach in evoking aspects of a student's concept image as well as eliciting learning events. The essential difference between what is usually done in textbook presentations and what was done in the interviews is the extent of the student-provided validation. In the usual textbook presentations, examples and counterexamples are presented and explained whereas in our study students were required to decide whether a candidate was an example without autoritative confirmation by an outside source. Utilizing the structure of our interview in presenting new concepts in the classroom may be a useful pedagogical tool, but further research is needed to determine if this method of presentation is effective and efficient.

NOTES

[1] The authors would like to thank the referees for their many helpful suggestions.

[2] In the U.S., students at the undergraduate level typically take courses in a particular subject area such as mathematics which constitute approximately one-third of their course work over four years. This course-work develops the subject in some depth and is referred to as the student's major. A third-year (respectively, fourth-year) undergraduate student is referred to as a junior (respectively, a senior).

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