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Enumeration of hamiltonian paths in Cayley diagrams

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Abstract. Let G be a group generated by a subset of elements S. The Cayley diagram of G given S is the labeled directed graph with vertices identified with the elements of G and (v, u) is an edge labeled h if $h \in S$ and $uh = v$. The sequence of elements of S corresponding to the edges transversed in a hamiltonian path (whose initial vertex is the identity) is called a group generating sequence (abbreviated *ggs*) in S.

In this paper a minimal upper bound for the number of *ggs*'s in a pair of generator elements for any two-generated group is given. For all groups of the form $G = \langle a, b: b^n = 1, a^m = b^r, ba = ab^{-1}\rangle$ where m is even, it is shown that the number of *ggs*'s in $\{a, b\}$ is $1 + m(n-1)/2$. An algorithm is developed that yields the number of *ggs*'s for two-generated groups $G = \langle a, b\rangle$ for which $\langle ba^{-1}\rangle \lhd G$. Explicit forms for the counted *ggs*'s are also provided.

1. Introduction

Let G be a group generated by a subset of elements S. The Cayley diagram of G given S, denoted $D_S(G)$, is the labeled directed graph with vertices identified with the elements of G and (v, u) an edge labeled h if $h \in S$ and $uh = v$.

The existence of hamiltonian paths and circuits in Cayley diagrams has been studied by several authors. Rankin [11, 12] has provided some sufficient and necessary conditions on the group and generating set. Holsztyński and Nathanson [see 4] showed that there exists a hamiltonian path in every Cayley diagram of an abelian or hamiltonian group or a group whose order is no more than 15. Witte [14] proved the same for groups having a cyclic subgroup of index 2. However, there are Cayley diagrams having no hamiltonian path. Nijenhuis and Wilf [10, pp. 288–289] gave the example of the symmetric group on five letters given the generating set {(1 2), (1 2 3 4 5)}. According to Nathanson [9], Milnor constructed a solvable group having no hamiltonian path.

Necessary and sufficient conditions for a hamiltonian circuit to exist in a

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Cayley diagram of a cyclic group were found by Holsztyński and Strube [4]. The same were found by Trotter and Erdös [13] for the Cayley diagram generated by (0, 1) and (1, 0) of the Cartesian product of any two cyclic groups. Klerlein and Starling [7] have generalized some of these results for semi-direct products of two cyclic groups. Witte [14] has shown that a hamiltonian circuit exists in every Cayley diagram of abelian p-groups, metacyclic p-groups, any p-group whose order is no more than p^4, and dihedral groups of order $2n$ when n is divisible by at most three distinct primes. Witte, Letzter and Gallian [15] and Letzter [8] found hamiltonian circuits for Cartesian products of a variety of Cayley diagrams. Keating [5] did the same for the conjunctive product. Curran [1] has begun to investigate Cartesian products of three cyclic groups.

Another question of interest is the following. Which Cayley diagrams defined by a minimal generating set have hamiltonian circuits? Klerlein [6] has answered this question in the affirmative for finite abelian groups, dihedral groups, and groups whose order is less than 16. Rankin [12] showed the same for any group with a two element generating set in which one of the elements is of order 2. Witte [14] has placed an upper bound on the cardinality of the smallest generating set of a group that yields a Cayley graph with a hamiltonian circuit.

The enumeration of hamiltonian paths and circuits has also been done. Nathanson [9] has shown that $D_G(G)$ has exactly $|G|!$ hamiltonian paths regardless of G. Gallian and Marttila [3] found the number of hamiltonian paths in Cayley diagrams of dihedral groups for various generating sets. This paper treats a class of groups that includes the dihedral and generalized quaternion groups. Gallian [2] also presented a variation of the problem in conjunction with an application to letter sorting. For all groups $G = \langle a, b\rangle$ in which $\langle ba^{-1}\rangle \lhd G$, Rankin [11] found the number of hamiltonian circuits. This paper determines the number of hamiltonian paths in Rankin's groups.

2. The general case

Let G be a finite group generated by a subset of elements S, and denote the order of G by N. Let y_i denote the i-th vertex of a hamiltonian path. The enumeration of hamiltonian paths in $D_S(G)$ can be simplified by considering only hamiltonian paths whose initial vertex is the identity. Indeed, if $y_0, y_1, \ldots, y_{N-1}$ is a hamiltonian path in $D_S(G)$, then $1, y_0^{-1}y_1, y_0^{-1}y_2, \ldots, y_0^{-1}y_{N-1}$ is also a hamiltonian path in $D_S(G)$. Conversely, if $1, y_1, y_2, \ldots, y_{N-1}$ is a hamiltonian path in $D_S(G)$, then $y, yy_1, yy_2, \ldots, yy_{N-1}$ is also a hamiltonian path in $D_S(G)$ for each $y \in G$. Hence, the number of hamiltonian paths is simply the product of the order of G and the number of hamiltonian paths whose initial vertex is the identity.

The sequence $x_1, x_2, \ldots, x_{N-1}$ of elements of S is called a *group generating sequence*, abbreviated *ggs*, if the partial products of the sequence $1, x_1, x_1x_2, \ldots, x_1x_2 \cdots x_{N-1}$ are distinct and hence contain every element of G. Clearly the partial products $y_i = x_1x_2 \cdots x_i$ and the x_i can be identified as the vertices and edges transversed, respectively, of a hamiltonian path $y_0, y_1, \ldots, y_{N-1}$ in $D_S(G)$ whose initial vertex is the identity ($y_0 = 1$). If this hamiltonian path is also a circuit, then the *ggs* is said to be *cyclic*. Thus, the enumeration of hamiltonian paths and circuits is equivalent to the enumeration of *ggs* and cyclic *ggs*, respectively. (But note that, while each *ggs* determines N distinct hamiltonian paths, the N hamiltonian circuits determined by a cyclic *ggs* are all the same circuit with different "initial" vertices.)

Only two-generated groups will be considered in this paper. So, let $G = \langle S \rangle$ and $S = \{a, b\}$. Again let N be the order of G and x_i and y_i be the i-th element and i-th partial product (i.e., $y_i = x_1x_2 \cdots x_i$), respectively, of a *ggs*. A partial product y_i of a *ggs* is said to *travel by a* (resp., *by b*) if $x_{i+1} = a$ (resp., $x_{i+1} = b$). The $(N-1)$-st partial product of a *ggs* (i.e., y_{N-1}) is called the *whole product* of the *ggs*. The left cosets of $\langle ba^{-1} \rangle$ are called *outbound cosets*. If C is an outbound coset, then $a^{-1} \in C$ if and only if $b^{-1} \in C$. The unique outbound coset containing both a^{-1} and b^{-1} is called *special;* all other outbound cosets are called *regular.* The motivation for considering these cosets will become apparent in the following proposition.

PROPOSITION. *Let C be an outbound coset which does not contain the whole product of some ggs. Let y_i, y_j be partial products of this ggs. If $y_i, y_j \in C$, then y_i travels by a if and only if y_j travels by a.*

Proof. We need to show that $x_{i+1} = a$ if and only if $x_{j+1} = a$. If $x_{i+1} = a$, then $y_{i+1} = y_ia$. Let $y_p = y_iab^{-1}$. There must be a $(p+1)$-st element in the *ggs* since y_p is contained in the same outbound coset as y_i, and this outbound coset does not contain the whole product. If $x_{p+1} = b$, then $y_{p+1} = y_i(ab^{-1})b = y_ia$ which cannot occur since a *ggs* has distinct partial products. Thus, $x_p = a$. Now let $y_p = y_i(ab^{-1})^k$. If the partial product $y_i(ab^{-1})^{k-1}$ travels by a, by the same argument as used before, $x_{p+1} = a$. Thus, $x_{j+1} = a$. Since the argument is symmetric in i and j, the converse also follows.

COROLLARY. *The whole product of a ggs must be an element of the special outbound coset.*

Proof. Suppose the special outbound coset does not contain the whole product. By the Proposition its elements travel either entirely by a or entirely by b.

In either case, $1 = a^{-1}a = b^{-1}b$ is a partial product. This contradiction establishes the Corollary.

These two results allow us to apply the term "travels by" to regular outbound cosets, that is, a regular outbound coset C is said to travel by a in some ggs if some partial product $y_i \in C$ travels by a. This makes it feasible for a computer to generate a large number of examples since one does not have to check all 2^{N-1} possible sequences for a group of order N.

THEOREM 1. *Let* $G = \langle a, b\rangle$, $s = |ba^{-1}|$, *and* $t = |G|/s$. *Then the number of ggs's of* G *is at most* $[(s+1)/2]2^{t-1}$, *where* [ ] *is the greatest integer function.*

Proof. In view of the Proposition and the above remark, a ggs is completely determined once we know (1) the element by which each regular outbound coset travels and (2) the integer w $(0 \le w \le s-1)$ such that the whole product is $a^{-1}(ba^{-1})^w$. Since there are at most 2^{t-1} possibilities for the elements in (1), and s possibilities for w, there can be at most $2^{t-1}s$ ggs's. To improve this estimate, we assert that once the elements in (1) have been chosen, no two consecutive values of w in (2) can both give rise to ggs's. Indeed, suppose that both $u = a^{-1}(ba^{-1})^w$ and $v = a^{-1}(ba^{-1})^{w+1}$ were whole products of ggs's (with the same choice of the elements in (1)). In these ggs's all elements except u and v travel the same way. This means that each ggs is an initial segment of the other, which is absurd. Now since the maximum number of non-consecutive integers in the interval $[0, s-1]$ is $[(s+1)/2]$, it follows that the number of ggs's is at most $[(s+1)/2]2^{t-1}$.

Theorem 1 cannot be improved for two-generated groups in general as the following example shows; however, other than the quaternion group of order 8 (see Theorem 3), the author knows of no two-generated non-abelian groups or groups with $|ba^{-1}| > 2$ for which the number of ggs's equals the maximum allowable by Theorem 1. Witte, in a private communication, has been able to obtain very restrictive conditions upon the groups that attain the maximum. So, it may be possible to strengthen the theorem for the "majority" of two-generated groups.

EXAMPLE. $G = \langle a, b : b^n = 1, a^2 = b^2, ba = ab\rangle$ where n is even. For G, $|ba^{-1}| = 2$ and $G = 2n$, so by Theorem 1 there are no more than 2^{n-1} ggs's. Consider all sequences of the form $x_1, x_2, \ldots, x_{2n-1}$ where $x_i = x_{n+1}$ for $i = 1, 2, \ldots, n-1$. Since $y_{n-1} = a^{-1}$ or b^{-1}, eliminate from consideration those sequences for which $y_n = 1$. The remaining 2^{n-1} sequences are ggs's because $1 \ne y_n = x_1x_2 \cdots x_n$ which implies that $x_1x_2 \cdots x_i \ne x_1x_2 \cdots x_{n+i}$ for $i = 0, 1, \ldots, n-1$ and $x_1x_2 \cdots x_i \ne x_1x_2 \cdots x_j$ if $|j - i| = n$, and so the partial products are distinct. Therefore, G has exactly 2^{n-1} ggs's.

3. Metacyclic groups

A group G is called *metacyclic* if it has a cyclic normal subgroup B such that G/B is cyclic. Let b generate B and $a \in G/B$ so that $G = \langle a, b\rangle$. Before enumerating ggs's (in a and b) of special classes of metacyclic groups, we develop a suitable characterization of all metacyclic groups.

Let the order of b be n. Since $\langle b\rangle \lhd G$, there exists a smallest positive integer $h < n$ such that

$$ba = ab^h. \tag{1}$$

By induction

$$b^j a^i = a^i b^{jh^i}. \tag{2}$$

Every product of powers of a and b may be reduced to the form $a^i b^j$ by repeated application of (2). If $m = |G|/n$, then

$$a^m = b^r \tag{3}$$

for some positive integer $r \le n$, and no power of a less than m is equal to any power of b (for otherwise the number of elements in the group would be less than $mn = |G|$). So, each element of G can be uniquely expressed in the form $a^i b^j$ with $0 \le i < m$ and $0 \le j < n$. Since $|b| = n$, (2) implies that

$$(h, n) = 1. \tag{4}$$

With $i = m$ and $j = 1$ in (2), equ. (3) implies that

$$h^m \equiv 1 \pmod{n}. \tag{5}$$

With $i = 1$ and $j = r$ in (2), equ. (3) implies that

$$r(h-1) \equiv 0 \pmod{n}. \tag{6}$$

Conversely, given positive integers n, m, $r \le n$, and $h < n$ satisfying (4)–(6),

$$G(n, m, r, h) = \langle a, b \colon b^n = 1, a^m = b^r, ba = ab^h\rangle \tag{7}$$

defines a metacyclic group of order mn. Indeed, (4) ensures that $|b| = n$, (5) and

(6) ensure associativity, and the remaining group axioms are trivially true. Of course, $G(n, m, r, h) \cong G(n', m', r', h')$ can occur with some of the unprimed and primed quantities not identical; however, in addition to the particular group under consideration, we are also interested in the particular generating set, that is, the Cayley diagram is the relevant object to consider. Two Cayley diagrams $D_S(G)$ and $D_T(H)$ are isomorphic if and only if there is a group isomorphism $f: G \to H$ such that $f(S) = T$. In the case at hand, it can be shown that the Cayley diagrams of $G(n, m, r, h)$ and $G(n', m', r', h')$ are isomorphic if and only if one of the following holds:

(1) $(n, m, r, h) = (n', m', r', h')$, or

(2) $m = r'$

$r = m'$

$mn = m'n'$

$m(h-1) + m'(h'-1) \equiv 0 \pmod{mn}$.

An example is $G(12, 2, 6, 7)$ and $G(4, 6, 2, 3)$. If one is concerned only with the number of ggs's and cyclic ggs's but not in their form, the relevant isomorphism class is all isomorphic *unlabeled* Cayley diagrams. For example, the groups $G(8, 1, 3, 1)$ and $G(4, 2, 2, 3)$ nor their Cayley diagrams are isomorphic; however, their unlabeled Cayley diagrams are isomorphic and so both have the same number of ggs's (4) and cyclic ggs's (2).

PROPOSITION. *The sequence* $\overbrace{\underbrace{b, \ldots, b}_{n-1}, a, \ldots, \underbrace{b, \ldots, b}_{n-1}, a}^{m-1}, \underbrace{b, \ldots, b}_{n-1}$ *is a ggs for every metacyclic group* $G = \langle a, b\rangle$, *where* $\langle b\rangle \lhd G$.

Proof. The k-th segment of the form $b, \ldots b, a$ has partial products of the form $a^{k-1}b^j$, for $j = 0, 1, \ldots, n-1$ (not necessarily in that order), and so are distinct. Since no power of a less than m is equal to any power of b, all partial products are distinct.

So, every metacyclic group $G = \langle a, b\rangle$, where $\langle b\rangle \lhd G$ has a ggs in a and b; however, the same cannot be said for cyclic ggs's for it can be shown that $G(12, 2, 10, 7)$ and $G(12, 2, 12, 7)$ have no cyclic ggs's as shown in Table 1. Table 1 gives the number of ggs's and cyclic ggs's in a and b for all non-abelian metacyclic groups $G = \langle a, b\rangle$, where $\langle b\rangle \lhd G$ and G has order 24. The author knows of no general algorithm for calculating the ggs's of metacyclic groups other than the one implied by the proof of Theorem 1; however, as noted in Table 1,

Table 1
The number of ggs's and cyclic ggs's in a and b for all non-abelian metacyclic groups G of order 24 as characterized by equation (7). The last column gives the theorem(s) used in each enumeration. The Cayley diagrams of $G(12, 2, 6, 7)$ and $G(4, 6, 2, 3)$ form the unique pair of groups listed whose Cayley diagrams, labeled or unlabeled, are isomorphic.

n	m	r	h	ggs's	cggs's	Theorem used
12	2	2	7	52	32	5
		3	5	12	3	1
		4	7	10	4	1
		6	5	600	600	1
			7	10	2	5
			11	12	6	3, 1
		8	7	1024	1024	1
		9	5	11	3	1
		10	7	8	0	5
		12	5	9	6	1
			7	4	0	1
			11	12	12	2, 1
6	4	3	5	11	3	3, 1
		6	5	11	6	2, 1
4	6	2	3	10	2	3, 1
		4	3	10	4	2, 1
3	8	3	2	9	3	2, 1

the enumeration can be simplified in certain special cases. We consider these in the final two sections.

4. A special class of metacyclic groups

In this section we shall deal with metacyclic groups for which $h=n-1$ (see definition (7) in Section 3). Equation (5) of the previous section implies that m must be even unless $n=2$. Equation (6) of the previous section implies that $n \mid 2r$ and since $r \leq n$, we have $n=r$ or $2r$. These groups turn out to be a generalization of two familiar classes of groups: the dihedral ($n=r$ and $m=2$) and the generalized quaternions ($n=2r$ and $m=2$). For this class of groups it turns out to be more useful to write elements in the form $b^i a^j$ where $0 \leq i < n$ and $0 \leq j < m$. The Cayley diagram for these groups is given in Figure 1. It will be helpful to the reader to keep this figure in mind while working through the proofs that follow.

If x_i and x_j are elements of a ggs and $i \leq k \leq j$ implies that $x_k = a$ if and only if $k=i$ or j, then x_i and x_j are said to be *consecutive* a's and $x_{i+1}, x_{i+2}, \ldots, x_{j-1}$ is

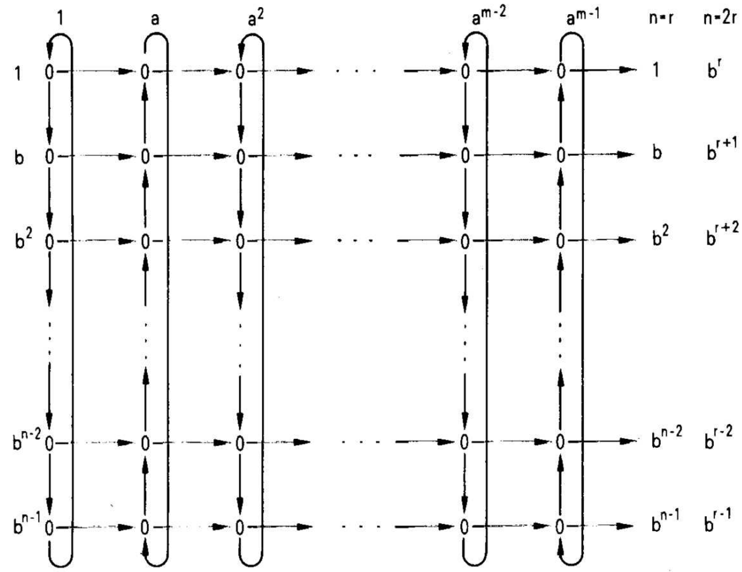

Figure 1
The Cayley diagram in $\{a, b\}$ of the metacyclic group $G = \langle a, b : b^n = 1,\ a^m = b^r,\ ab = b^{-1}a\rangle$. The horizontal and vertical directed line segments correspond to the elements a and b, respectively, of a sequence. The vertices adjacent from $b^i a^{m-1}$ for $i = 0, 1, \ldots, n-1$ depend upon whether $n = r$ or $n = 2r$ as indicated.

called a *b-segment.* The number $j-i-1$ (which may be zero) is called the b-segment's *length.* One b-segment immediately follows or immediately precedes another b-segment when there is exactly one a between them; hence, in the sequence segment $a, \underbrace{b, \ldots, b}, a, a, \underbrace{b, \ldots, b}, a$ the two b-segments in brackets do not immediately follow or precede each other; however, a b-segment of zero length does immediately follow the first bracketed b-segment and immediately precedes the second bracketed b-segment. As before, y_j will denote $x_1 x_2 \cdots x_j$.

LEMMA 1. *In a ggs the length of any b-segment must be at least as great as the length of the b-segment immediately preceding it unless the product of the latter b-segment is* b^{-1}. *(See Figure* 2.)

Proof. Let x_i, x_j and x_k be three consecutive a's in a ggs with $j-i > k-j$. If the element $y_{k-1}b$ is not the identity, then there must exist a partial product y_p such that $y_p a = y_{k-1}b$ or $y_p b = y_{k-1}b$. Since $x_k = a$, the latter is not possible. If the former were true, then $y_p = y_{k-1}ba^{-1} = y_i(b^{j-i-1}ab^{k-j-1})ba^{-1} = y_i b^{2j-i-k-1}$. But $i \le 2j-k-1 \le j-2$ so that $y_i b^{2j-i-k-1}$ already travels by b. Thus, $y_{k-1}b = 1$.

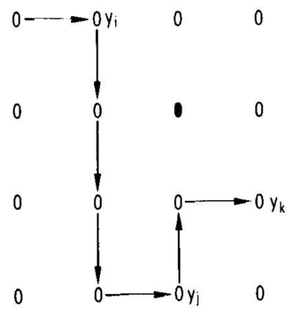

Figure 2
An impossible path by Lemma 1 since ● cannot be a partial product.

LEMMA 2. *In a ggs only one b-segment can have length greater than the length of the b-segment immediately preceding it.* (*See Figure* 3.)

Proof. Let x_i, x_j and x_k be three consecutive a's in a ggs with $j-i<k-j$. Consider the element $y_p=y_ib^{-1}$. Now $y_i=y_pb$ and $y_{2j-i}=y_ib^{j-i-1}ab^{j-i}=y_ib^{-1}a=y_pa$. But $x_i=a$ and $x_{2j-i}=b$, so y_p must be the whole product. If there were two pairs of sequences as described in the lemma, the ggs would have to have two different whole products which is absurd.

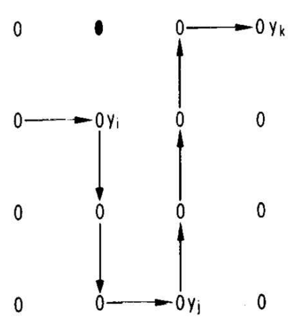

Figure 3
Only one such path exists by Lemma 2 since ● must be the whole product.

THEOREM 2. *Suppose G is a metacyclic group with* $h=n-1$, $n=r$, *and* m *even. Then G has* $1+m(n-1)/2$ ggs's. *These ggs's are of the form*

$$\overbrace{\underbrace{b,\ldots,b}_{n-1}, a, \ldots, \underbrace{b,\ldots,b}_{n-1}, a}^{m-1}, \underbrace{b,\ldots,b}_{n-1}; \quad \text{or}$$

$$\overbrace{\underbrace{b,\ldots,b}_{p}, a, \ldots, \underbrace{b,\ldots,b}_{p}, a}^{2i-1}, \overbrace{\underbrace{b,\ldots,b}_{n-1}, a, \ldots, \underbrace{b,\ldots,b}_{n-1}, a}^{m-2i+1},$$

$$\overbrace{\underbrace{b,\ldots,b}_{n-2-p}, a, \ldots, \underbrace{b,\ldots,b}_{n-2-p}, a}^{2i-2}, \underbrace{b,\ldots,b}_{n-2-p}$$

where $p=0, 1, \ldots, n-2$ *and* $i=1, 2, \ldots, m/2$.

Proof. (In this proof and in the proof of Theorem 3, len (i) will denote the length of the i-th b-segment, and y_u will denote the u-th partial product where u should be clear from context. For example, the first time the notation is used below, $u=(p+1)(2i)+(q+1)(m-2i)$.) After a bit of calculation or working with Figure 1, one can ascertain that the above sequences are indeed *ggs*'s. To show that these are the only *ggs*'s, Lemmas 1 and 2 require us to consider only the following two cases.

CASE 1. If $\text{len}\,(1)=\text{len}\,(2)=\cdots=\text{len}\,(2i)=p\le q=\text{len}\,(2i+1)=\text{len}\,(2i+2)=\cdots=\text{len}\,(m)$, then $y_u=(b^pa)^{2i}(b^qa)^{m-2i}=1$; hence, $mn\le u=2i(p-q)+m(q+1)\le m(q+1)$ which implies that $n-1\le q$. Since the partial products of a *ggs* are distinct, $q\le n-1$ also holds; hence $q=n-1$. Substituting this value for q in the original inequality, we obtain $0\le 2i(p-n+1)\le 0$ which implies that $p=n-1$. Therefore, the first *ggs* listed in the theorem is the only possible *ggs*.

CASE 2. If $\text{len}\,(1)=\text{len}\,(2)=\cdots=\text{len}\,(2i-1)=p<q=\text{len}\,(2i)=\text{len}\,(2i+1)=\cdots=\text{len}\,(m)$, then $y_u=(b^pa)^{2i-1}(b^qa)^{m-2i+1}=b^{p-q}$. Now $s=\text{len}\,(m+1)<q-p$ for otherwise 1 would be a partial product twice, and $\text{len}\,(m+2)=\text{len}\,(m+3)=\cdots=\text{len}\,(m+2i-1)=s$ by Lemma 2. Now consider $y_v=y_u(b^sa)^{2i-1}=b^{p-q+s}a^{2i-1}$ where $p-q\le j\le -1$. But $y_w=(b^pa)^{2i-1}b^t=b^{p-t}a^{2i-1}$ where $0\le t\le q$ (or we can say $p-q\le p-t\le p$) are already partial products. Since the partial products of a *ggs* are distinct, $mn\le v=(2i-1)(p-q+s+1)+m(q+1)\le m(q+1)$ which implies that $q\ge n-1$, and therefore $q=n-1$. Substituting this value for q in the inequality in the last sentence, we obtain $0\le p-(n-1)+s+1\le 0$ or $s=n-2-p$. Thus, only those sequences listed in the theorem are possible *ggs*'s.

THEOREM 3. *Suppose G is a metacyclic group with $h = n-1$, $n = 2r$, and m even. Then G has $1 + m(n-1)/2$ ggs's. These ggs's are of the form*

$$\overbrace{\underbrace{b,\ldots,b}_{n-1},a,\ldots,\underbrace{b,\ldots,b}_{n-1},a}^{m-1},\underbrace{b,\ldots,b}_{n-1};$$

$$\overbrace{\underbrace{b,\ldots,b}_{r-1},a,\ldots,\underbrace{b,\ldots,b}_{r-1},a}^{2j},\overbrace{\underbrace{b,\ldots,b}_{n-1},a,\ldots,\underbrace{b,\ldots,b}_{n-1},a}^{m-2j},$$
$$\overbrace{\underbrace{b,\ldots,b}_{r-1},a,\ldots,\underbrace{b,\ldots,b}_{r-1},a}^{2j-1},\underbrace{b,\ldots,b}_{r-1};$$

$$\overbrace{\underbrace{b,\ldots,b}_{p},a,\ldots,\underbrace{b,\ldots,b}_{p},a}^{2i-1},\overbrace{\underbrace{b,\ldots,b}_{r-1},a,\ldots,\underbrace{b,\ldots,b}_{r-1},a}^{2m-2i+1},$$
$$\overbrace{\underbrace{b,\ldots,b}_{r-2-p},a,\ldots,\underbrace{b,\ldots,b}_{r-2-p},a}^{2i-2},\underbrace{b,\ldots,b}_{r-2-p};\quad \textit{or}$$

$$\overbrace{\underbrace{b,\ldots,b}_{p},a,\ldots,\underbrace{b,\ldots,b}_{p},a}^{m+2i-1},\overbrace{\underbrace{b,\ldots,b}_{r-1},a,\ldots,\underbrace{b,\ldots,b}_{r-1},a}^{m-2i+1}$$
$$\overbrace{\underbrace{b,\ldots,b}_{r-2-p},a,\ldots,\underbrace{b,\ldots,b}_{r-2-p},a}^{m+2i-2},\underbrace{b,\ldots,b}_{r-2-p}$$

where $j = 0, 1, \ldots, m/2-1$, $p = 0, 1, \ldots, r-2$, and $i = 1, 2, \ldots, m/2$.

Proof. After a bit of calculation or working with Figure 1, one can see that the above sequences are indeed ggs's. The proof to show that these sequences are the only possible ggs's is best approached by considering the following five cases.

CASE 1. If $\text{len}(1) = \cdots = \text{len}(2i) = p < q = \text{len}(2i+1) = \cdots = \text{len}(m)$, then $y_u = (b^p a)^{2i}(b^q a)^{m-2i} = b^r$. If $p \geq r$, then $mn \leq u = 2i(p-q) + m(q+1) < m(q+1)$ which implies that $q > n-1$. But then the partial products of the sequence would not all be distinct, so $p < r$. If $q < r$, then let $\text{len}(m+1) = \cdots = \text{len}(2m) = s$. This implies that $s < r$ and $y_v = y_u(b^s a)^m = 1$. So, $v = 2i(p-q) + (q+s+2)m < -1 + (q+s+2)m \leq nm - 1$; hence, some element of G is not a partial product of the sequence, and so $q \geq r$. Let $\text{len}(m+1) = \cdots = \text{len}(m+2i) = s$. This implies that $s < r$ and $y_v = y_u(b^s a)^{2i} = b^r a^{2i} = (b^p a)^{2i} b^r$ is already a partial product; hence, $mn \leq v = 2i(p-q+s+1) + m(q+1) \leq m(p+r+1)$. So, $p = r-1$. Substitution of this value for p back in the inequality yields $mn \leq 2i(r-q+s) + m(q+1) \leq m(r+s+1)$, so that $s = r-1$ and $q = n-1$. Thus, the second set of ggs's (less when $j = 0$) are the only possible.

CASE 2. If $\text{len}\ (1)=\cdots=\text{len}\ (2i-1)=p\leq q=\text{len}\ (2i)=\cdots=\text{len}\ (m)$ and $r\leq p<n$, then $y_u=(b^pa)^{2i-1}(b^qa)^{m-2i+1}=b^{p-q+r}$; hence, $mn\leq u=(2i-1)(p-q)+m(q+1)\leq m(q+1)$ which implies that $q\geq n-1$, and hence $q=n-1$. Substitution of this value for q in the above inequality yields $0\leq p-n+1\leq 0$ or $p=n-1$. Thus, the first *ggs* listed in the theorem is the only possible one.

CASE 3. If $\text{len}\,(1)=\cdots=\text{len}\,(2i-1)=p\leq q=\text{len}\,(2i)=\cdots=\text{len}\,(m)$ and $p=r-1$, then $y_u=(b^pa)^{2i-1}(b^qa)^{m-2i+1}=b^{p-q+r}$. Unless $p=q$, the sequence will not have distinct partial products. Let $\text{len}\,(m+1)=\cdots=\text{len}\,(m+k)=s\leq t=\text{len}\,(m+k+1)=\cdots=\text{len}\,(2m)$. This implies that $y_v=y_u(b^sa)^k(b^ta)^{m-k}=1$ (if k is even) and b^{r+s-t} (if k is odd). Either way the partial products are not distinct unless $mn\leq v=k(s-t)+m(r+t+1)\leq m(r+t+1)$ which implies that $t\geq r-1$. But for the partial products to be all distinct, $t\leq r-1$; hence, $t=r-1$. Substitution of this value for t in the above inequality yields $0\leq s-r+1\leq 0$ or $s=r-1$. Thus, the *ggs* in the second set listed in the theorem with $j=0$ is the only one possible.

CASE 4. If $\text{len}\,(1)=\cdots=\text{len}\,(2i-1)=p<q=\text{len}\,(2i)=\cdots=\text{len}\,(m)$ and $0\leq p<r-1$, then $y_u=(b^pa)^{2i-1}(b^qa)^{m-2i+1}=b^{p-q+r}=b^e$ where $e=p-q+r$ if $p-q+r>0$ and $e=p-q+r+n$ if $p-q+r<0$. Since the partial products must be distinct, $p<e<n$. Let $\text{len}\,(m+1)=\cdots=\text{len}\,(2m)=s$. Then $y_v=y_u(b^sa)^{2i-1}=b^{e+s}a^{2i-1}$; however, the sequence already has partial products of the form $y_{z+j}=(b^pa)^{2i-1}b^j=b^{p-j}a^{2i-1}$ for $0\leq j\leq q$ which implies that $e+s<n+p-q$. If $e=p-q+r+n$, then this last inequality implies that $r+s<0$ which is impossible; hence, e can only equal $p-q+r$. Substitution of this into the inequalities $p<e$ and $e+s<n+p-q$ yields $q<r$ and $s<r$, respectively.

Let $\text{len}\,(2m+1)=\cdots=\text{len}\,(2m+2i-1)=t$. Then $y_{w+k}=y_u(b^sa)^mb^k=b^{p-q+k}$ implies that $n+p-q+k<n$ for $0\leq k\leq t$. Also $y_x=y_w(b^ta)^{2i-1}=b^{p-q+t}a^{2i-1}=(b^pa)^{2i-1}b^{q-t}$ is a partial product already because $0\leq t<q$. Therefore, $mn\leq x=(2i-1)(p-q+t+1)+m(q+s+2)\leq m(q+s+2)$ which implies that $q+s\geq n-2$. To avoid repeating partial products, $q+s=n-2$. Substitution of this equality into the inequality of the preceding sentence yields $p-q+t+1=0$. Since $q<r$ and $s<r$, we obtain that $q=s=r-1$. Finally, rearrangement of $p-q+t+1=0$ yields $t=r-2-p$. Thus, the third form of *ggs* listed in the theorem is the only possible form.

CASE 5. If $\text{len}\,(1)=\cdots=\text{len}\,(m)=p<r-1$, then $y_{u+q}=(b^pa)^mb^q=b^{r+q}$ where $q=\text{len}\,(m+1)$ which implies that $r+q<n$ or $q<r$. Suppose $\text{len}\,(m+1)=\cdots=\text{len}\,(m+2i)=q\leq s=\text{len}\,(m+2i+1)=\cdots=\text{len}\,(2m)$. Then $y_v=y_u(b^qa)^{2i}(b^sa)^{m-2i}=1$ which implies that $mn\leq v=2i(q-s)+m(p+s+2)$;

hence, $p+s \geq n-2$. But this is impossible since $p<r-1$ and $s \leq r-1$. Thus, we must have $\text{len}(m+1) = \cdots = \text{len}(m+2i-1) = q < s = \text{len}(m+2i) = \cdots = \text{len}(2m)$. So, $y_v = y_u(b^q a)^{2i-1}(b^s a)^{m-2i+1} = b^{q-s}$ and $y_{v-1} = b^{q-s+r}a^{m-1}$ imply that $r+q<q-s+n$ or $s<r$ and $p<q-s+r$, respectively.

If $\text{len}(2m+1) = \cdots = \text{len}(3m) = t$, then $y_{v+j} = y_v b^j = b^{q-s+j}$ for $0 \leq j \leq t$ which implies that $q-s+t+n<n$. If $\text{len}(3m+1) = \cdots = \text{len}(3m+2i-1) = d$, then $y_{w+d} = y_v(b^t a)^m b^d = b^{q-s+r+d}$, but $y_u = b^r$ must not be repeated so $q-s+r+d<r$. Also $y_z = y_w(b^d a)^{2i-1} = b^{q-s+r+d}a^{2i-1} = b^e a^{2i-1}$ where $q-s+r \leq e < r$, but there are already partial products of the form $y_u(b^q a)^{2i-1}b^j = b^{q-j+r}a^{2i-1}$ where $0 \leq j \leq s$ which implies that $q-s+r \leq q-j+r \leq r+q$. Thus, $mn \leq z$.

Now consider the powers of b as partial products: $y_k = b^j$ for $k = 0, 1, \ldots, p$; $w, w+1, \ldots, w+d$; $u, u+1, \ldots, u+q$; $v, v+1, \ldots, v+t$ with $j = 0, 1, \ldots, p$; $q-s+r, q-s+r+1, \ldots, q-s+r+d$; $r, r+1, \ldots, r+q$; $n+q-s, n+q-s+1, \ldots, n+q-s+t$. Since these are the only powers of b appearing as partial products, the following four equalities must hold: $p+1 = q-s+r$, $q-s+r+d+1 = r$, $r+q+1 = n+q-s$, and $n+q-s+t+1 = n$. Solving these equations simultaneously, we find that $p = q$, $s = r-1$, and $t = d = r-2-p$. Thus, the fourth form of ggs listed in the theorem is the only possible form.

THEOREM 4. *Suppose G is a metacyclic group with $n = 2$. If $r = 2$, then G has $[m/2]+1$ ggs's. If $r = 1$, then G has $[(m+1)/2]+1$ ggs's. These ggs's are of the form*

$$\underbrace{a, \ldots, a}_{m-i}, \underbrace{b, a, \ldots, b, a}_{i+1}, \underbrace{a, \ldots, a}_{m-i-2}$$

where $i = m$ and the odd positive integers less than m if $r = 2$ and $i = 0, m$, and the even positive integers less than m if $r = 1$.

The proof is clear from the Cayley diagram of G.

5. Rankin's groups

Let $G^* = \langle a, b \rangle$ and $\langle ba^{-1} \rangle < G^*$. Such groups were studied by Rankin [8], hence the designation in this section's title. These are metacyclic groups, so they can be characterized in the same fashion as in Section 3:

$$G^*(N, M, R, H) = \langle a, b : c^N = 1, a^M = c^R, ca = ac^H \rangle, \qquad (1)$$

where $c = ba^{-1}$, is such a group if and only if there exist positive integers $N, M, R \leq N$, and $H < N$ such that

$$(H, N) = 1$$
$$H^M \equiv 1 \pmod N$$
$$R(H-1) \equiv 0 \pmod N.$$

Even though these are metacyclic groups, we are not considering a subclass of the Cayley diagrams focused upon in Section 3 because $\langle b \rangle$ may not be normal in G^*. Nonetheless, these two classes of groups do overlap. $G^*(N, M, R, H)$ has the property that a and $b = ca$ generate a group of the form $G(n, m, r, h)$ if and only if $(N, \psi) \mid H^2 - H$ where $\psi = R + 1 + H + \cdots + H^{M-1}$. In this case,

$$m = (N, \psi)$$
$$n = M\frac{N}{m}$$
$$h = 1 + Mk, \qquad k \equiv \left(\frac{H^2 - H}{m}\right)\left(\frac{\psi}{m}\right)^{-1} \left(\mathrm{mod}\, \frac{N}{m}\right)$$
$$r = m + Mk', \qquad k' \equiv \left(\frac{H + \cdots + H^m}{m}\right)\left(\frac{\psi}{m}\right)^{-1} \left(\mathrm{mod}\, \frac{N}{m}\right).$$

Again let $c = ba^{-1}$. Our assumption that $\langle c \rangle$ be normal in G^* imposes a simple structure upon the outbound cosets and their interrelationship. The outbound cosets are clearly of the form $a^i\langle c \rangle$ for $0 \leq i < M$, $a^i\langle c \rangle = a^j\langle c \rangle$ if and only if $i \equiv j \pmod M$, and $a^{M-1}\langle c \rangle$ is the special outbound coset. Now

$$a^i c^j a = a^{i+1} c^{jH} \tag{2}$$
$$a^i c^j b = a^{i+1} c^{(j+1)H}.$$

Hence, if the partial product of a ggs $y_k \in a^i\langle c \rangle$, then $y_{k+1} \in a^{i+1}\langle c \rangle$. So, the Cayley diagram of the generators $\{a, b\}$ in G^* is an M-partite graph whose components are connected in cyclical fashion, and any ggs must be of the form

$$S, x_M, S, x_{2M}, \ldots, S, x_{(N-1)M}, S \tag{3}$$

where the choice of the whole product determines x_{jM} for $j = 1, 2, \ldots, N-1$ and

S is the subsequence $x_1, x_2, \ldots, x_{M-1}$ where $a^i\langle c\rangle$ travels by x_{i+1} for $i = 0, 1, \ldots, M-2$.

We wish to determine when a sequence of the form (3) is a *ggs*, or equivalently, determine when the M-partite Cayley graph has a hamiltonian path whose initial vertex is the identity. Since $a = c^0a$ and $b = c^1a$,

$$x_1x_2 \cdots x_{M-1} = a^{M-1}c^s \tag{4}$$

where

$$s \equiv \sum_{i=1}^{M-1} e_iH^{M-1} \pmod N$$

$$e_i = \begin{cases} 0, & \text{if } x_i = a \\ 1, & \text{if } x_i = b \end{cases}.$$

Using (2) and the same argument as in the proof of Theorem 1, we see that if a^ic^j travels by a in a *ggs*, then a^ic^{j-1} travels by a in the *ggs* unless it is the whole product. Similarly, if a^ic^j travels by b in a *ggs*, then a^ic^{j+1} travels by b in the *ggs* unless it is the whole product. Also $b^{-1} = a^{M-1}c^{-R-1}$ and $a^{-1} = a^{M-1}c^{-R}$ travel by a and b, respectively, unless either is the whole product. Hence, given that the whole product is $a^{M-1}c^w$, then $a^{M-1}c^j$ travels by a if $w + R < j + R \leq N-1$ and travels by b if $0 \leq j + R < w + R$, where all quantities are taken modulo N. Finally note that $(a^{M-1}c^j)(c^ea) = c^{(j+e)H+R}$.

For a given special segment and whole product the M-partite Cayley graph can now be reduced to a bipartite graph with N vertices in each component. If the vertices in each component are labeled from 0 to $N-1$ consecutively, vertex j of the first component is adjacent to vertex $s + jH^{M-1} \pmod N$ of the second component where s is determined by (4). Vertex $j \neq w$ of the second component is adjacent to vertex $(j+e)H + R \pmod N$ where

$$e = \begin{cases} 0, & \text{if } w + R < j + R \leq N - 1 \pmod N \\ 1, & \text{if } 0 \leq j + R < w + R \pmod N \end{cases}$$

and vertex w is adjacent to no vertex. Thus, the original sequence is a *ggs* if and only if the above obtained bipartite graph has a hamiltonian path whose initial vertex is vertex 0 in the first component and terminal vertex is vertex w in the second component, or equivalently, the obtained graph is connected. Clearly, this criterion can be further reduced to a question of the connectedness of a graph consisting of N vertices. This is the essence of the following theorem.

THEOREM 5. *Suppose G^* is a group generated by the elements a and b with the subgroup generated by $c = ba^{-1}$ normal in G^*. Let N, M, R, and H be as defined in* (1). *Then a sequence of a's and b's is a ggs if and only if the sequence is of the form* (3) *and* 0, $f(0)$, $f^{(2)}(0), \ldots, f^{(N-1)}(0)$ *are all defined and distinct* ($f^{(i)}$ *denotes f composed with itself i times*) *where*

$$f(j) \equiv \begin{cases} j+R+Hs, & \text{if } w+R<t_j \le N-1 \\ j+R+H(s+1), & \text{if } 0 \le t_j < w+R \\ \text{undefined}, & \text{if } t_j \equiv w+R \end{cases} \tag{5}$$

where all operations are modulo N, $t_j \equiv jH^{M-1}+R+s \pmod N$, *and s is defined by* (4).

The number of ggs for a given group is given by

$$\sum_{s=0}^{N-1} \alpha(s)\beta(s) \tag{6}$$

where $\alpha(s)$ is the number of $(M-1)$-tuples $(e_1, e_2, \ldots, e_{M-1})$ of 0's and 1's for which $\sum_{i=1}^{M-1} e_i H^{M-i} = s$ and $\beta(s)$ is the number of whole products w, $0 \le w < N$, for which s and w satisfy the criterion for a ggs by the theorem. A computer can be readily used to compute (6) as long as the order of G is not too large. Some calculations have been performed by the author and are presented in Table 2.

Since a ggs is cyclic if and only if the whole product is a^{-1} or b^{-1} ($w = -R$ or $-R-1$), the theorem immediately implies the following result due originally to Rankin.

COROLLARY. (Rankin [8, Theorem 4]). *Suppose G^* is as in the theorem. Let S be the subsequence $x_1, x_2, \ldots, x_M$ where $x_i = c^{e_i}a$ for $i = 1, 2, \ldots, M$; let $t \equiv R + \sum_{i=1}^{M} e_i H^{M+1-i} \pmod N$. Then a sequence is a cyclic ggs if and only if $(t, N) = 1$ and the sequence will be of the form S, S, ..., S.*

The results simplify somewhat when the special case of abelian groups ($H = 1$) is considered. For this case (5) becomes

$$f(j) \equiv \begin{cases} t_j, & \text{if } w+R<t_j \le N-1 \\ t_j+1, & \text{if } 0 \le t_j < w+R \\ \text{undefined}, & \text{if } t_j \equiv w+R, \end{cases}$$

Table 2
The number of ggs's and cyclic ggs's in a and b for some abelian Rankin groups G^* as characterized by (1) of order 24.

N	M	R	ggs's	cggs's
12	2	8	8	0
		9	10	1
		10	7	2
8	3	1	13	4
		2	12	4
6	4	1	18	2
		2	18	4
		3	15	7
4	6	1	44	32
		2	48	32
		3	52	32
3	8	2	170	170
		3	171	171
2	12	1	2048	2048

where all operations are modulo $N, t_j \equiv j+R+s \pmod N$, and $s \equiv \sum_{i=1}^{M-1} e_i \pmod N$. So, since s is just the number of b's $\pmod N$ in segment S of (3), to calculate the number of ggs, the following formula can be used in (6):

$$\alpha(s) = \sum_{\substack{j=0 \\ j \equiv s \pmod N}}^{M-1} \binom{M-1}{j}.$$

The same considerations in conjunction with the corollary yield upon simplification the following formula for the number of cyclic ggs:

$$\sum_{\substack{j=0 \\ (R+j,N)=1}}^{M} \binom{M}{j}.$$

The results of this section are not quite as "nice" as those of Section 4, but it does not appear that the results can be simplified (see Table 2) except in more restrictive subclasses.

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