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*Note*

# Core and monotonic allocation methods

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**Abstract.** Young showed in a paper of 1985 (*Int. J. Game Theory* **14**, 65–72) that no core allocation method can be coalitionally monotonic on cooperative games with five or more players. This note extends Young's result. No core allocation method can be coalitionally monotonic on cooperative games with four or more players, and there is an infinite class of core allocation methods that are coalitionally monotonic on three-player cooperative games. *Journal of Economic Literature*

**Key words:** Value, cooperative game, allocation, axiomatic

An *n-player cooperative game* is a real-valued function $v$ defined on all *coalitions* $S \subseteq N = \{1, 2, \ldots, n\}$ such that $v(\emptyset) = 0$. An *allocation* for the $n$-player game $v$ is a real number vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ satisfying $\sum_{i \in N} x_i = v(N)$. The latter condition is referred to as *efficiency*. An *allocation method* is a function $\theta$ which assigns for each game an allocation. The *core* of a game $v$, denoted by Core($v$), is the set of all allocations $\mathbf{x}$ satisfying $\sum_{i \in S} x_i \geq v(S)$ for all coalitions $S \subseteq N$. An allocation method $\theta$ is a *core allocation method* if $\theta(v) \in \text{Core}(v)$ whenever $\text{Core}(v) \neq \emptyset$. An allocation method $\theta$ is *coalitionally monotonic* if an increase in the worth of a particular coalition implies no decrease in the allocation to any member of that coalition: $u(T) \leq v(T)$ for some coalition $T$ and $u(S) = v(S)$ for all coalitions $S \neq T$ implies $\theta_i(u) \leq \theta_i(v)$ for all $i \in T$. Both properties are desirable: no coalition can do better

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on its own if a core allocation method is used, and there is no incentive to underreport worths of coalitions if a coalitionally monotonic allocation method is used.

Young (1985) proved there exists *no* core allocation method that is coalitionally monotonic on games with five or more players. This result is reported again in Young (1994). We extend this result to games with four players in Theorem 1 and show that the result cannot be extended to games with three players in Theorem 3. In the interest of notational brevity, we will dispense with set brackets and commas when writing specific coalitions, for example, $v(\{1,2\})$ and $\{\{1,2\},\{1,3\},\{2,3\}\}$ will be written $v(12)$ and $\{12,13,23\}$, respectively.

**Theorem 1.** *For $|N| = 4$ there is no core allocation method that is coalitionally monotonic.*

*Proof.* Suppose $\theta$ is a core allocation method that is coalitionally monotonic on four-player games. Consider the four-player game $v$ defined by $v(1) = v(2) = v(3) = v(4) = v(12) = v(34) = 0$, $v(N) = 2$, and $v(S) = 1$ otherwise. Consider the four different games $v^1$, $v^2$, $v^3$, and $v^4$, which are defined to be the same as $v$ except that $v^1(134) = 2$, $v^2(234) = 2$, $v^3(123) = 2$, and $v^4(124) = 2$. In each new game, the core has a unique allocation: $\mathrm{Core}(v^1) = \mathrm{Core}(v^2) = \{(0,0,1,1)\}$ and $\mathrm{Core}(v^3) = \mathrm{Core}(v^4) = \{(1,1,0,0)\}$.[1] Since $\theta$ is a core allocation method, $\theta(v^1) = \theta(v^2) = (0,0,1,1)$ and $\theta(v^3) = \theta(v^4) = (1,1,0,0)$.

Notice that $v(134) < v^1(134)$ and $v(S) = v^1(S)$ for all $S \neq \{1,3,4\}$. Since $\theta$ is coalitionally monotonic, $\theta_1(v) \leq \theta_1(v^1) = 0$. By similar arguments, we can show that $\theta_i(v) \leq \theta_i(v^i) = 0$ for $i = 1,2,3,4$. But this violates efficiency of the allocation $\theta(v)$. Thus, $\theta$ cannot be both core and coalitionally monotonic. $\square$

**Definition.** *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ be a vector consisting of positive real numbers. Define the $\alpha$-excess of a nontrivial (not $\varnothing$ or $N$) coalition $S$ with respect to the allocation $\mathbf{x}$ to be $e_\alpha(S,\mathbf{x}) = \alpha_{|S|}(v(S) - \sum_{i\in S} x_i)$ where $|S|$ is the number of players in $S$. Let $e_\alpha(\mathbf{x})$ be the vector of $\alpha$-excesses ordered from largest to smallest. The* **$\alpha$-prenucleolus** *of the game $v$ is the allocation that lexicographically minimizes $e_\alpha(\cdot)$.*

The $\alpha$-prenucleolus is a generalization of two better known allocation methods. The nucleolus defined by Schmeidler (1969) has $\alpha_k = 1$ for all $k$. The per capita nucleolus defined by Grotte (1970) has $\alpha_k = 1/k$ for all $k$. Geometrically (see the Figure), the $\alpha$-prenucleolus is the lexicographic center of the core: If the core is nonempty, it is shrunk by moving each coalition hyperplane $\sum_{i\in S} x_i = v(S) + c_S$ by increasing each $c_S$ from zero at a rate proportional to the reciprocal of $\alpha_{|S|}$. Movement is stopped when further movement would result in an empty set. If a unique allocation is not obtained, the coalition hyperplanes not forming the boundary of the shrunk core con-

[1] Indeed, it is easily verified that the given allocations are contained in the stated cores. Now suppose $x \in \mathrm{Core}(v^1)$. Then $0 = v^1(2) \leq x_2 = v^1(N) - x_1 - x_3 - x_4 \leq v^1(N) - v^1(134) = 0$, and so $x_2 = 0$. Furthermore, $x_3 = x_2 + x_3 \geq v^1(23) = 1$ and $x_4 = x_2 + x_4 \geq v^1(24) = 1$. It now follows from efficiency and $x_1 \geq 0$ that $x_3 = x_4 = 1$. We conclude that $\mathrm{Core}(v^1) = \{(0,0,1,1)\}$. A similar argument can be given for each of the remaining games.

tinue to move until further movement would result in an empty set. This process is repeated until a unique allocation is obtained. If the core is empty, the initial movement of the coalition hyperplanes is outward (decreasing the $c_S$) until a nonempty set is obtained.

*Example.* Consider the game $v$ defined by $v(123) = 2$, $v(23) = 1$, and $v(S) = 0$ otherwise. Suppose $\alpha_1 = 1 - t$, $\alpha_2 = t$, and $0 < t < 2/3$. We will show that $\mathbf{x} = (t, 1 - t/2, 1 - t/2)$ is the $\alpha$-prenucleolus of $v$. Note first that $e_\alpha(1, \mathbf{x}) = e_\alpha(23, \mathbf{x}) = -t(1 - t)$, $e_\alpha(12, \mathbf{x}) = e_\alpha(13, \mathbf{x}) = -t(1 + t/2) \leq -t(1 - t)$, and $e_\alpha(2, \mathbf{x}) = e_\alpha(3, \mathbf{x}) = -(1 - t)(1 - t/2) \leq -t(1 - t)$ where the inequalities follow from the restriction $0 < t < 2/3$. Now suppose that $\mathbf{y}$ is an allocation for which $e_\alpha(\mathbf{y})$ is no larger lexicographically than $e_\alpha(\mathbf{x})$. So, the largest excess of $\mathbf{y}$ can be no larger than the largest excess of $\mathbf{x}$. In particular, $(1-t)(-y_1) = e_\alpha(1, \mathbf{y}) \leq -t(1-t)$ and $t(1 - y_2 - y_3) = e_\alpha(23, \mathbf{y}) \leq -t(1-t)$. So, $y_1 \geq t$ and $y_2 + y_3 \geq 2 - t$. By efficiency, $2 = y_1 + y_2 + y_3 \geq 2$. Hence, the earlier inequalities must be equalities. In particular, $y_1 = t = x_1$. A similar argument involving the next largest excesses of $\mathbf{x}$ and $\mathbf{y}$ shows that $\mathbf{y} = \mathbf{x}$. Hence, $\mathbf{x}$ is the allocation that lexicographically minimizes $e_\alpha(\cdot)$. Geometrically (see Figure 1), the $x_1 = c_1$ and $x_2 + x_3 = 2 + c_{23}$ hyperplanes meet first.

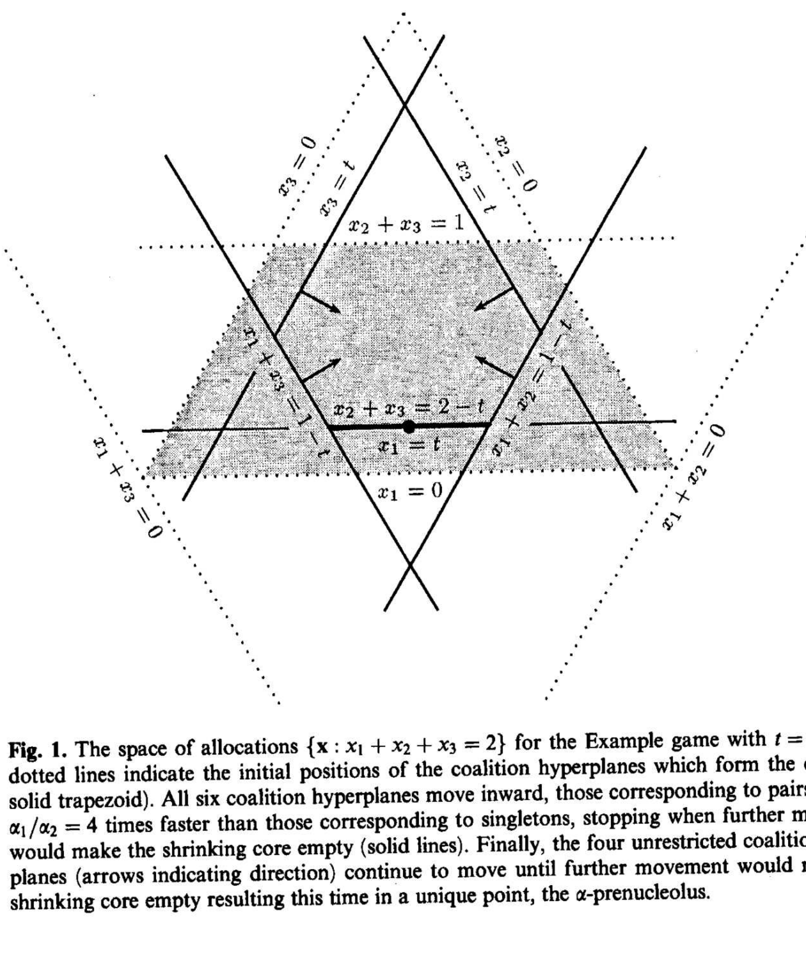


**Fig. 1.** The space of allocations $\{\mathbf{x} : x_1 + x_2 + x_3 = 2\}$ for the Example game with $t = 0.2$. The dotted lines indicate the initial positions of the coalition hyperplanes which form the core (the solid trapezoid). All six coalition hyperplanes move inward, those corresponding to pairs moving $\alpha_1/\alpha_2 = 4$ times faster than those corresponding to singletons, stopping when further movement would make the shrinking core empty (solid lines). Finally, the four unrestricted coalition hyperplanes (arrows indicating direction) continue to move until further movement would make the shrinking core empty resulting this time in a unique point, the $\alpha$-prenucleolus.

Next either (1) the $x_1 + x_2 = c_{12}$ and $x_1 + x_3 = c_{13}$ hyperplanes meet if $t \leq 2/5$, or (2) the $x_2 = c_2$ and $x_3 = c_3$ hyperplanes meet if $t \geq 2/5$. The $\alpha$-prenucleolus is the unique intersection of the hyperplanes.

**Theorem 2.** *The $\alpha$-prenucleolus is a core allocation method.*

*Proof.* Suppose there exists an allocation $\mathbf{y} \in \text{Core}(v)$. Then $\sum_{i \in S} y_i \geq v(S)$ for all coalitions $S$. Since $\alpha_{|S|} > 0$, $e_\alpha(S, \mathbf{y}) = \alpha_{|S|}(v(S) - \sum_{i \in S} y_i) \leq 0$ for all coalitions $S$. Since the $\alpha$-prenucleolus $\mathbf{x}$ of $v$ lexicographically minimizes $e_\alpha(\cdot)$ among all allocations, $\alpha_{|S|}(v(S) - \sum_{i \in S} x_i) = e_\alpha(S, \mathbf{x}) \leq 0$ for all coalitions $S$. Since $\alpha_{|S|} > 0$, $\sum_{i \in S} x_i \geq v(S)$ for all coalitions $S$. Hence, $\mathbf{x} \in \text{Core}(v)$. □

**Theorem 3.** *For $|N| = 3$ the $\alpha$-prenucleoli satisfying $\alpha_1 \geq \alpha_2 > 0$ form an infinite class of core allocation methods which are coalitionally monotonic.*

*Proof.* The Example (with the restriction $0 < t \leq 1/2$) and Theorem 2 show that the $\alpha$-prenucleoli satisfying $\alpha_1 \geq \alpha_2 > 0$ is an infinite class of core allocation methods. To complete the proof, we will show that for all $\alpha_1 \geq \alpha_2 > 0$, the $\alpha$-prenucleolus is coalitionally monotonic. We will do this by an explicit construction and examination of all possible formulas for the $\alpha$-prenucleoli.

Suppose $\mathbf{x}$ is the $\alpha$-prenucleolus for the 3-player game $v$. Let $\mathscr{B}_1$ be the collection of all nontrivial coalitions with maximum coalitional excess. We claim that $\mathscr{B}_1$ must be a superset of $\{12, 13, 23\}$ or a partition of $N$. Indeed, if $\mathscr{B}_1$ is not a superset of $\{12, 13, 23\}$ nor a partition of $N$, then $\mathscr{B}_1$ must consist of either

1. one pair $\{ij\}$;
2. two pairs $\{ij, jk\}$;
3. one singleton or one singleton with pairs that include the singleton $\{j\}$, $\{j, ij\}$, or $\{j, ij, jk\}$; or
4. two singletons or two singletons with their union $\{i, j\}$ or $\{i, j, ij\}$.

In the first and last cases, let $y_i = x_i + \varepsilon$, $y_j = x_j + \varepsilon$, and $y_k = x_k - 2\varepsilon$. In the middle two cases, let $y_j = x_j + 2\varepsilon$, $y_i = x_i - \varepsilon$, and $y_k = x_k - \varepsilon$. Clearly, $\mathbf{y}$ is an allocation. For sufficiently small $\varepsilon > 0$, $e_\alpha(\mathbf{y})$ will be lexicographically smaller than $e_\alpha(\mathbf{x})$. This contradiction to the assumption that $\mathbf{x}$ is the $\alpha$-prenucleolus verifies the claim.[2]

By the verified claim, we need consider only three cases: $\mathscr{B}_1$ is a superset of $\{12, 13, 23\}$, $\{1, 2, 3\}$, or a partition of the players into a singleton and a pair, which we will denote $\{1, 23\}$ by possibly relabeling the players.

*Case 1.* Suppose $\{2, 13, 23\} \subseteq \mathscr{B}_1$. Then $e_\alpha(12, \mathbf{x}) = e_\alpha(13, \mathbf{x}) = e_\alpha(23, \mathbf{x})$. With the addition of the efficiency condition, $x_1 + x_2 + x_3 = v(123)$, this system of equations has the unique solution

$$x_1 = \tfrac{1}{3}(v(123) + v(12) + v(13) - 2v(23))$$

$$x_2 = \tfrac{1}{3}(v(123) + v(12) + v(23) - 2v(13))$$

$$x_3 = \tfrac{1}{3}(v(123) + v(13) + v(23) - 2v(12)).$$

[2] Note that we have essentially proved a special case of Kohlberg's (1971) characterization of the nucleolus (generalized here to the $\alpha$-prenucleolus).

*Case 2*. Suppose $\{1,2,3\} \subseteq \mathscr{B}_1$. Then $e_\alpha(1,\mathbf{x}) = e_\alpha(2,\mathbf{x}) = e_\alpha(3,\mathbf{x})$. With the addition of the efficiency condition, this system of equations has the unique solution

$$x_1 = \tfrac{1}{3}(v(123) + 2v(1) - v(2) - v(3))$$
$$x_2 = \tfrac{1}{3}(v(123) + 2v(2) - v(3) - v(1))$$
$$x_3 = \tfrac{1}{3}(v(123) + 2v(3) - v(1) - v(2)).$$

*Case 3*. Suppose $\{1,23\} \subseteq \mathscr{B}_1$. Then $e_\alpha(1,\mathbf{x}) = e_\alpha(23,\mathbf{x})$ and the efficiency condition yield two equations. Let $\mathscr{B}_2$ be the collection of all nontrivial coalitions not in $\mathscr{B}_1$ with maximal coalitional excess. We claim that $\mathscr{B}_2$ must be a superset of $\{12,13\}$, $\{2,3\}$, or a partition of $N$. Indeed, if $\mathscr{B}_2$ is not a superset of $\{12,13\}$, $\{2,3\}$, nor a partition of $N$, then $\mathscr{B}_2$ must be $\{j\}$, $\{1j\}$, or $\{j,1j\}$ where $j=2$ or $j=3$. Let $y_1 = x_1$, $y_j = x_j + \varepsilon$, and $y_k = x_k - \varepsilon$. Clearly, $\mathbf{y}$ is an allocation satisfying $e_\alpha(1,\mathbf{y}) = e_\alpha(23,\mathbf{y}) = e_\alpha(1,\mathbf{x}) = e_\alpha(23,\mathbf{x})$. For sufficiently small $\varepsilon > 0$, $e_\alpha(\mathbf{y})$ will be lexicographically smaller than $e_\alpha(\mathbf{x})$. This contradiction to the assumption that $\mathbf{x}$ is the $\alpha$-prenucleolus verifies the claim.

By the verified claim, we need consider only three subcases: $\mathscr{B}_2$ is a superset of $\{12,13\}$, $\{2,3\}$, or a partition of the players into a singleton and a pair, which we will denote $\{2,13\}$ by possibly relabeling the players. Each subset yields a third independent equation. Let $\hat{\alpha}_k = \alpha_k/(\alpha_1 + \alpha_2)$ for $k = 1,2$.

*Case 3a*. Suppose $\{12,13\} \subseteq \mathscr{B}_2$. Then $e_\alpha(12,\mathbf{x}) = e_\alpha(13,\mathbf{x})$. With the addition of the Case 3 equations, this system has the unique solution

$$x_1 = \hat{\alpha}_2 v(123) - \hat{\alpha}_2 v(23) + \hat{\alpha}_1 v(1)$$
$$x_2 = \tfrac{1}{2}(\hat{\alpha}_1 v(123) - v(13) + v(12) + \hat{\alpha}_2 v(23) - \hat{\alpha}_1 v(1))$$
$$x_3 = \tfrac{1}{2}(\hat{\alpha}_1 v(123) - v(12) + v(13) + \hat{\alpha}_2 v(23) - \hat{\alpha}_1 v(1)).$$

*Case 3b*. Suppose $\{2,3\} \subseteq \mathscr{B}_2$. Then $e_\alpha(2,\mathbf{x}) = e_\alpha(3,\mathbf{x})$. With the addition of the Case 3 equations, this system has the unique solution

$$x_1 = \hat{\alpha}_2 v(123) - \hat{\alpha}_2 v(23) + \hat{\alpha}_1 v(1)$$
$$x_2 = \tfrac{1}{2}(\hat{\alpha}_1 v(123) + \hat{\alpha}_2 v(23) - \hat{\alpha}_1 v(1) + v(2) - v(3))$$
$$x_3 = \tfrac{1}{2}(\hat{\alpha}_1 v(123) + \hat{\alpha}_2 v(23) - \hat{\alpha}_1 v(1) - v(2) + v(3)).$$

*Case 3c*. Suppose $\{2,13\} \subseteq \mathscr{B}_2$. Then $e_\alpha(2,\mathbf{x}) = e_\alpha(13,\mathbf{x})$. With the addition of the Case 3 equations, this system has the unique solution

$$x_1 = \hat{\alpha}_2 v(123) - \hat{\alpha}_2 v(23) + \hat{\alpha}_1 v(1)$$
$$x_2 = \hat{\alpha}_2 v(123) - \hat{\alpha}_2 v(13) + \hat{\alpha}_1 v(2)$$
$$x_3 = (\hat{\alpha}_1 - \hat{\alpha}_2) v(123) + \hat{\alpha}_2 v(13) + \hat{\alpha}_2 v(23) - \hat{\alpha}_1 v(1) - \hat{\alpha}_1 v(2).$$

Note that each set of formulas holds on a full-dimensional, closed, and convex subset of the vector space of all three-player games.[3] Further, where two or more sets of formulas are applicable, they must agree with each other because the coalitional excess equalities defining each set of formulas must all be satisfied.

An examination of each formula for $x_i$ shows that the coefficients of $v(S)$ for all coalitions $S$ containing $i$ are nonnegative (we use our assumption that $\alpha_1 \geq \alpha_2$ in the formula for $x_3$ in case 3c). Hence, at every game $v$, the $\alpha$-prenucleolus for player $i$ is a nondecreasing function of $v(S)$ for each coalition $S$ containing $i$. Thus, the $\alpha$-prenucleolus is coalitionally monotonic on the space of three-player games. □

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[3] For example, consider the Case 1 set of formulas. They must yield the $\alpha$-prenucleolus whenever these formulas yield $e_\alpha(12, \mathbf{x}) = e_\alpha(13, \mathbf{x}) = e_\alpha(23, \mathbf{x}) \geq e_\alpha(i, \mathbf{x})$ for all players $i$. After some algebraic manipulation, we obtain the equivalent conditions $2v(ij) + 2v(ik) + v(jk) \geq v(123) + v(i)$ for all players $i$.