

Note

Core and monotonic allocation methods

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Abstract. Young showed in a paper of 1985 (*Int. J. Game Theory* 14, 65–72) that no core allocation method can be coalitionally monotonic on cooperative games with five or more players. This note extends Young's result. No core allocation method can be coalitionally monotonic on cooperative games with four or more players, and there is an infinite class of core allocation methods that are coalitionally monotonic on three-player cooperative games. *Journal of Economic Literature*

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An *n*-player cooperative game is a real-valued function v defined on all coalitions $S \subseteq N = \{1, 2, \dots, n\}$ such that $v(\emptyset) = 0$. An allocation for the *n*-player game v is a real number vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ satisfying $\sum_{i \in N} x_i = v(N)$. The latter condition is referred to as *efficiency*. An allocation method is a function θ which assigns for each game an allocation. The core of a game v , denoted by $\text{Core}(v)$, is the set of all allocations \mathbf{x} satisfying $\sum_{i \in S} x_i \geq v(S)$ for all coalitions $S \subseteq N$. An allocation method θ is a core allocation method if $\theta(v) \in \text{Core}(v)$ whenever $\text{Core}(v) \neq \emptyset$. An allocation method θ is coalitionally monotonic if an increase in the worth of a particular coalition implies no decrease in the allocation to any member of that coalition: $u(T) \leq v(T)$ for some coalition T and $u(S) = v(S)$ for all coalitions $S \neq T$ implies $\theta_i(u) \leq \theta_i(v)$ for all $i \in T$. Both properties are desirable: no coalition can do better

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on its own if a core allocation method is used, and there is no incentive to underreport worths of coalitions if a coalitionally monotonic allocation method is used.

Young (1985) proved there exists *no* core allocation method that is coalitionally monotonic on games with five or more players. This result is reported again in Young (1994). We extend this result to games with four players in Theorem 1 and show that the result cannot be extended to games with three players in Theorem 3. In the interest of notational brevity, we will dispense with set brackets and commas when writing specific coalitions, for example, $v(\{1, 2\})$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ will be written $v(12)$ and $\{12, 13, 23\}$, respectively.

Theorem 1. *For $|N| = 4$ there is no core allocation method that is coalitionally monotonic.*

Proof. Suppose θ is a core allocation method that is coalitionally monotonic on four-player games. Consider the four-player game v defined by $v(1) = v(2) = v(3) = v(4) = v(12) = v(34) = 0$, $v(N) = 2$, and $v(S) = 1$ otherwise. Consider the four different games v^1 , v^2 , v^3 , and v^4 , which are defined to be the same as v except that $v^1(134) = 2$, $v^2(234) = 2$, $v^3(123) = 2$, and $v^4(124) = 2$. In each new game, the core has a unique allocation: $\text{Core}(v^1) = \text{Core}(v^2) = \{(0, 0, 1, 1)\}$ and $\text{Core}(v^3) = \text{Core}(v^4) = \{(1, 1, 0, 0)\}$.¹ Since θ is a core allocation method, $\theta(v^1) = \theta(v^2) = (0, 0, 1, 1)$ and $\theta(v^3) = \theta(v^4) = (1, 1, 0, 0)$.

Notice that $v(134) < v^1(134)$ and $v(S) = v^1(S)$ for all $S \neq \{1, 3, 4\}$. Since θ is coalitionally monotonic, $\theta_1(v) \leq \theta_1(v^1) = 0$. By similar arguments, we can show that $\theta_i(v) \leq \theta_i(v^i) = 0$ for $i = 1, 2, 3, 4$. But this violates efficiency of the allocation $\theta(v)$. Thus, θ cannot be both core and coalitionally monotonic. \square

Definition. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ be a vector consisting of positive real numbers. Define the α -excess of a nontrivial (not \emptyset or N) coalition S with respect to the allocation \mathbf{x} to be $e_\alpha(S, \mathbf{x}) = \alpha_{|S|}(v(S) - \sum_{i \in S} x_i)$ where $|S|$ is the number of players in S . Let $e_\alpha(\mathbf{x})$ be the vector of α -excesses ordered from largest to smallest. The α -prenucleolus of the game v is the allocation that lexicographically minimizes $e_\alpha(\cdot)$.

The α -prenucleolus is a generalization of two better known allocation methods. The nucleolus defined by Schmeidler (1969) has $\alpha_k = 1$ for all k . The per capita nucleolus defined by Grotte (1970) has $\alpha_k = 1/k$ for all k . Geometrically (see the Figure), the α -prenucleolus is the lexicographic center of the core: If the core is nonempty, it is shrunk by moving each coalition hyperplane $\sum_{i \in S} x_i = v(S) + c_S$ by increasing each c_S from zero at a rate proportional to the reciprocal of $\alpha_{|S|}$. Movement is stopped when further movement would result in an empty set. If a unique allocation is not obtained, the coalition hyperplanes not forming the boundary of the shrunk core con-

¹ Indeed, it is easily verified that the given allocations are contained in the stated cores. Now suppose $x \in \text{Core}(v^1)$. Then $0 = v^1(2) \leq x_2 = v^1(N) - x_1 - x_3 - x_4 \leq v^1(N) - v^1(134) = 0$, and so $x_2 = 0$. Furthermore, $x_3 = x_2 + x_3 \geq v^1(23) = 1$ and $x_4 = x_2 + x_4 \geq v^1(24) = 1$. It now follows from efficiency and $x_1 \geq 0$ that $x_3 = x_4 = 1$. We conclude that $\text{Core}(v^1) = \{(0, 0, 1, 1)\}$. A similar argument can be given for each of the remaining games.

