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APPORTIONMENT

Reflections on the Politics of Mathematics

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Dividing up the seats in the federal House of Representatives among fifty states might seem to be an elementary exercise in arithmetic, requiring merely a simple rounding of fractions. This is not the case, however. How to apportion the Congressional seats has been debated frequently and at length for over two hundred years, and the issue is likely to receive considerable attention by the Congress and the courts in the next year or two, in the wake of the recent national census.

The long-standing contention is over the method used to arrive at an integral number of representatives for each state from a mathematical quotient of population figures. In fact, the mathematical problem was the occasion of the first presidential veto in United States history: George Washington vetoed the initial apportionment bill passed by the nation's first Congress. Over the years, four different mathematical procedures have been followed and others have been proposed. Although all of them appear reasonable in their approach, none of those yet devised has all the "common-sense" properties any "reasonable" method should have. Also, the possibilities for political advantage give the controversy intriguing overtones that are unusual in debates about mathematics. In this article, we touch on the ramifications of the apportionment problem, as well as its mathematical basis.

We note that the problem of apportioning Congressional seats is distinct from the reapportionment or redistricting problem, which is concerned with determining the boundaries of the Congressional districts within a state. This problem is also one of major concern for many legislatures and courts this year.

THE HISTORICAL ROLE OF THE CENSUS

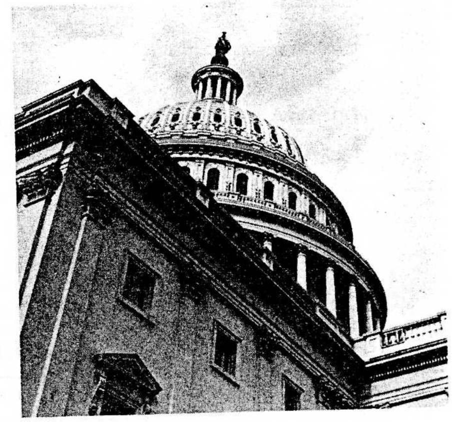
The apportionment issue was addressed in the United States Constitution, which established the census and set forth the first rules for enumeration and for allocation of taxes and Congressional seats.

Article I, Section 2 of the original Constitution includes the statement:

> *Representatives and direct taxes shall be apportioned among the several states which may be included within this Union, according to their respective numbers, which shall be determined by adding to the whole number of free persons, including those bound to service for a term of years, and excluding Indians not taxed, three-fifths of all other persons. The actual enumeration shall be made within three years after the first meeting of the Congress of the United States, and within every subsequent term of ten years, in such manner as they shall by law direct. The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one. . .*

The original stipulations about what groups should be counted have been modified a number of times. In 1868 the Fourteenth Amendment to the Constitution eliminated the three-fifths rule for counting slaves. The census of 1940 included all Indians. Since 1970 certain overseas citizens have been allocated to their "home"

"Representatives . . . are not finely divisible . . ."



states. And in 1980, in a controversial ruling, illegal aliens were included in the population counts.

The apportionment of taxes has also been revised. The Sixteenth Amendment, instituted in 1913, stated that taxes could be levied "without apportionment among the several states, and without regard to any census or numeration," and this cleared the way for direct taxes such as the personal income tax. Still, certain federal payments to the states and local communities are currently determined by population figures, so that a decrease in relative population size can cause a state to lose revenue-sharing funds as well as political representation.

There is, of course, no difficulty in allocating taxes or subsidies in such a way that they are directly proportional to population, because money is a finely divisible commodity. Representatives, however, are not finely divisible, and the Constitution does not specify a precise rule for determining the necessary whole numbers. This is the cause of the political apportionment problem.

THE MATHEMATICAL BASIS OF APPORTIONMENT METHODS

The fair or proportional share of representatives for a particular state i in a representative assembly of s states and h seats is given by its *quota*, q_i. This is related to the number of Congressional seats and the population figures according to the expression

$$q_i/p_i = h/p$$

where p_i is the population of state i and p (the sum $p_1 + p_2 + \ldots + p_s$) is the population of the whole country. Another way of stating the relation is that each q_i is proportional to the corresponding population p_i, with h/p the constant of proportionality. The apportionment problem consists of rounding the s fractions q_i to nearby integers a_i in the best or most equitable manner while preserving the relation

$$a_1 + a_2 + \ldots + a_s = h.$$

Since 1912, the House of Representatives has used the fixed house size of $h = 435$. Previous to that time, the size of the House was a variable, determined along with the particular scheme for making the actual apportionment.

At first glance, it would appear that the apportionment problem should involve merely some simple approximation technique for rounding fractions, or perhaps some natural measure of inequity that could be minimized. Actually, every procedure that has been proposed has undesirable properties, as we shall see in the following brief review of the methods and some of the consequences of their application.

HAMILTON'S METHOD OF LARGEST FRACTIONS

Given the integers h and $p_1, p_2, \ldots, p_s$, one natural way to solve the apportionment problem is to assign to each state the largest whole number $[q_i]$ contained in its quota q_i and then to assign any remaining seats, one each, to those states having the largest fractions $q_i - [q_i]$, until all h seats have been allocated.

Table I illustrates this apportionment method, using as an example a university consisting of four colleges with enrollments of 4,799, 2,934, 1,332, and 935 students and a joint student senate consisting of $h = 100$

Table I. APPORTIONMENTS FOR A UNIVERSITY SENATE

College	Number of Students	Quota	Integral Part	Fractional Part	Apportionment of Seats by Various Methods				
i	p_i	q_i	$[q_i]$	$q_i - [q_i]$	Hamilton	Jefferson	Adams	Webster	Hill
1	4,799	47.99	47	0.99	48	49	47	48	48
2	2,934	29.34	29	0.34	29	29	29	30	30
3	1,332	13.32	13	0.32	13	13	14	13	13
4	935	9.35	9	0.35	10	9	10	9	9
Totals	10,000	100	98	2.00	100	100	100	100	100
A Feasible Divisor x						97.9	102.3	99.0	99.0

members. The mathematical quota q_i for each college contains a whole number $[q_i]$, which is assigned as a number of seats in the senate; then the remaining seats (two) are assigned to the first and fourth colleges because they have the two largest fractional parts. The resulting apportionment is 48, 29, 13, and 10 seats.

This procedure is referred to as the method of *largest fractions* or of *greatest remainders*, and in Congressional history as the method of *Hamilton* or of *Vinton*. In March of 1792 the Congress passed the nation's first apportionment bill, stipulating that this scheme, as proposed by Alexander Hamilton, should be used to allocate 120 seats among the existing fifteen states. George Washington, on the advice of Thomas Jefferson, vetoed that bill, but the method was revived sixty years later and used until 1910. During that period, the procedure was generally referred to as the Vinton method of 1850, after Ohio Congressman Samuel F. Vinton. Several procedural variations were actually used, and political squabbling sometimes led to apportionments that were inconsistent with the basic principles of the method.

Some undesirable properties of this method can be seen in terms of our example of the university student senate (Table I). If we assume that a few students have changed colleges and that the new population figures are 4,797, 2,937, 1,330, and 936, Hamilton's method of largest fractions results in the apportionment of 48, 30, 13, and 9. One thing that has happened is that although the fourth college's quota has increased (from 9.35 to 9.36), it has lost a seat in the senate. It turns out that this particular "fault" is present in every "reasonable" apportionment method, being inherent in the discreteness of the mathematics involved. But there is something else undesirable happening here that does not occur in many other apportionment methods: the *relative* population increase in college 4 (1 part in 936) is greater than for college 2 (3 parts in 2,937), and yet college 4 loses a seat to college 2. This phenomenon is referred to as the *population paradox*.

An additional example illustrates another undesirable property. If the size of the university senate is increased from $h = 100$ to $h = 101$ seats, but the population figures remain as given in Table I, the quotas are calculated as 48.47, 29.63, 13.45, and 9.44, leading to the apportionment of 49, 30, 13, and 9. Even though no change in the populations has taken place, and the total number of seats has increased by one, college 4 has lost a seat. This is referred to as the *paradox of house size*, or in Congressional history as the *Alabama paradox*. It actually appeared in some of the apportionments being considered for the federal House of Representatives—in Rhode Island in the 1870s, in Alabama in the 1880s, and in Maine and Colorado in the 1900s—and in the latter two cases it caused great controversy.

There are many other apportionment methods that never involve the population or Alabama paradoxes; these include the divisor methods discussed below. On the other hand, the method of Hamilton does always possess one highly desirable attribute, called the *quota property*: the apportionment for each state is either the largest whole number contained in the quota or else the next highest integer. This quota property does not hold, in general, for any of the other methods described here.

DIVISOR METHODS INCLUDING JEFFERSON'S

In the early years there was a strong tendency for Congressmen to think in terms of an ideal population size for Congressional districts. They would fix on some size x called a *divisor* or *ratio*, and then examine the resulting *quotient* p_i/x in order to determine how many seats state i should have. The Constitution requires that x be at least 30,000, and the first apportionment bill, passed in 1792, used the value $x =$ 33,000. The value of x was normally arrived at before the house size h was determined. (This is similar to deciding on an average class size for a school district, and then determining the number of teachers needed.)

This approach, too, leads to a problem of rounding, since the populations of states are usually not integer multiples of x. Depending on what rule is used for rounding remainders, different variations of the divisor method of apportionment result.

The first formal method used to apportion the House of Representatives was a divisor method identified by the terms *greater divisors*, *rejected fractions*, or *assumed ratios*; it has also been referred to as the *Jefferson* or the *Seaton* method. (C. W. Seaton of the census office reproposed the method in 1881 after the Alabama paradox arose with the use of Hamilton's method.) This was the procedure followed by the Congress after the censuses of 1790 through 1830. It resolves the remainder problem simply by rejecting the fractional parts in p_i/x. If the house size h is fixed in advance, then it is necessary to solve for an x which satisfies the requirement that the total number of apportioned seats be equal to h. Also, there must be a rule for breaking ties in the unlikely event that they should occur. In our student senate example (Table I), this method apportions $h = 100$ into 49, 29, 13, and 9 seats, respectively, with $x = 97.9$. (If h were increased to 101 in this example, the apportionment would be the same as that obtained according to the largest fractions method of Hamilton.)

One difficulty with this method of Jefferson is the fact that it might give a state one or more full seats in excess of its quota. This occurs in our Table I example: college 1 is assigned 49 seats, although its quota is calculated as 47.99. This violation has actually occurred in Congressional apportionments; for example, in the 1820s and 1830s New York had 34 and 40 seats, although its quotas were only 32.5 and 38.6, respectively. Another undesirable feature of the Jefferson method is that it shows a statistical bias in favor of large states; this is because dropping fractions hurts small states more than large ones. That fact probably did not escape Jefferson and his friends from Virginia, the largest state at the time. If measured against the apportionment that Hamilton's method would have produced, Jefferson's method effectively took one seat away from Delaware and gave it to Virginia. Large districts in a representative assembly often have enough votes to pass an apportionment law that is advantageous to them.

In 1832 John Quincy Adams proposed another divisor method that is referred to as the method of *smallest division* or the *Adams* method. In this procedure, each state i is assigned the smallest whole number that is larger than or equal to its quotient p_i/x. This method rounds all fractions up to the next integer value, rather than rounding them down, as in the method of Jefferson. The effect on our university senate apportionment is shown in Table I. In some cases this method may assign to a state an apportionment that is lower than its quota (for example, New York's quota in the 1830s was 38.6, but Adams' method would have assigned it only 37 seats). Also, this method is biased in favor of small states; Adams was concerned about the shift in Congressional seats away from Massachusetts and New England as the United States population expanded toward the west. His proposal was never adopted by the Congress, however.

Still another divisor method was proposed in 1832 by Daniel Webster. Called the *Webster* method or the method of *major fractions*, it takes the middle road between the approaches of Jefferson and Adams. Each state's quotient p_i/x is rounded to the nearest integer. Theoretically, this method could violate both the lower and the upper quota conditions; that is, it could give rise to apportionments less than $[q_i]$, the integral part of the quota (see Table I), or to apportionments greater than $[q_i] + 1$. In practice, however, such a violation occurs only rarely. Furthermore, the Webster method has

The Congressional apportionment method championed for some seventy years by Cornell's redoubtable Walter F. Willcox has strong support again today, after forty years of eclipse.

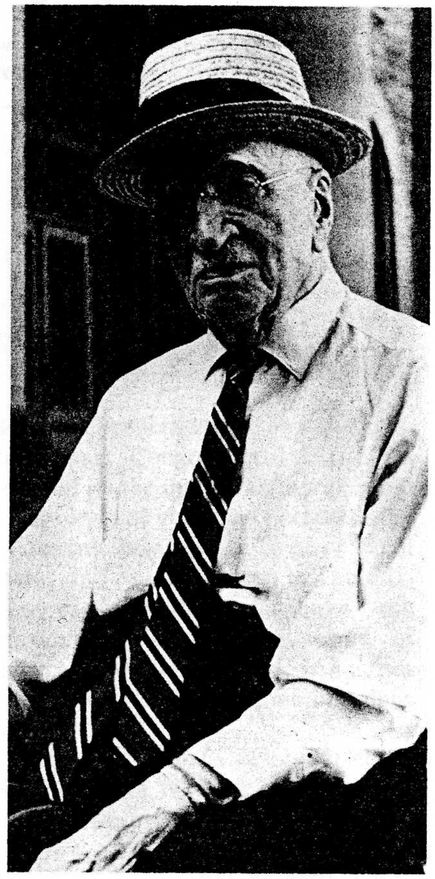
The eminent Willcox, who lived to the age of 103 (1861–1964) was known as a scholar with intellectual breadth and an interest in practical problems. An expert in the fields of statistics and demography, he was instrumental in introducing quantitative methods into social science research; as chief statistician for methods and research in the new Bureau of the Census (formed in 1900), he treated aspects such as illiteracy, race, and age, and issued reports that were the first of their kind and became models for ensuing studies. During his years at Cornell, from 1891 until his retirement in 1931, he also served as president of professional societies in several fields.

As a crusader for equitable Congressional apportionment, he was instrumental in establishing a limit of 435 members in the House (before this limit, the number increased with population). His advocacy of the method of major fractions for apportionment continued long into his retirement years; a letter on the subject, written when he was more than one hundred years old, was published in the New York Times.

Willcox was famous also as a walker. When he was in his nineties he was still walking at least six miles a day, and he walked at least a mile a day up to the time of his death.

Willcox came to Cornell in 1891 as a faculty member in philosophy; later, until his retirement in 1931, he was a professor of economics and statistics. He was dean of the College of Arts and Sciences from 1902 to 1907 and he became one of the first three faculty members on the Board of Trustees. He founded the Statler Club and served as its first president.

A son, Bertram F. Willcox, also taught at Cornell. He is an emeritus professor of law.



the advantage of being the only divisor method that does not favor either large states or small ones. This method of Webster was used by Congress after the 1840 census, and again from 1912 to 1940. It was replaced in 1852 by Hamilton's method of largest fractions (although several of the subsequent apportionments did not, in fact, follow the Hamilton procedure strictly, and some of them were in agreement with Webster's approach).

The major advocate for the Webster method throughout much of the first half of the twentieth century was the distinguished Cornell professor Walter F. Willcox. Although his position lost out after the 1940 census, the method he championed may yet be adopted.

THE CURRENT METHOD AND ITS LIABILITIES

In 1911 Joseph A. Hill of the Bureau of the Census suggested the apportionment method that was adopted following the census of 1940 and is still in effect. The reapportionment for the election in 1982 will also use the Hill method unless Congress acts to change the existing law.

The *Hill* method is also called the *main Huntington method* after Edward V. Huntington, a professor of mathematics and mechanics at Harvard, who was a major advocate of the method from 1920 through the early 1940s. Its basic principle is reflected in some of the other names attached to it: *equal proportions, geometric mean,* and *alternate ratios*. Both Hill and Huntington believed that an apportionment method should minimize some measure of inequity between pairs of states. The measure they chose was the *relative* (or percentage) difference between the values for average representation (the number of representatives per inhabitant) pertaining to the two states. Other measures, as Huntington showed, can lead to other apportionment methods. For example, if the *absolute* difference between the average representations of the two states were used, the method that would result is Webster's. Jefferson's and Adams' methods also minimize different pairwise measures of inequity.

Table II. PROPERTIES OF FIVE APPORTIONMENT METHODS				
Method of:	**Quota Violations**	**Alabama Paradox**	**Population Paradox**	**Bias toward:**
Adams	lower only	no	no	small states
Hill	yes	no	no	small states
Hamilton	no	yes	yes	neither
Webster	yes (few)	no	no	neither
Jefferson	upper only	no	no	large states

Hill's method is actually a divisor method, and so has the advantage of always avoiding the population and Alabama paradoxes, but it has the drawback of yielding apportionments that may be less than the lower or greater than the upper quota conditions. It is also somewhat biased in favor of small states.

The Hill method was voted into law by a Democratic Congress in 1942 when it was determined that this apportionment procedure would give an extra seat to Arkansas (normally a Democratic state), whereas Webster's method would give the seat to Michigan (which was likely to elect a Republican). The vote was strictly along party lines except for the Democrats from Michigan.

ASSESSING THE VARIOUS APPORTIONMENT METHODS

It has been proven that no apportionment method will simultaneously always avoid the population paradox and always produce an apportionment without quota violations. A choice must be made as to which of these two desirable properties is to be sacrificed. In the past, Congressional debate has focused on the undesirability of quota violations, the Alabama paradox, and systematic bias toward large or small states, but as can be seen in Table II, none of the methods that have been introduced eliminated all three of these. On the other hand, it has not been proven that such a method does not exist.

Until this issue is resolved, either the method of Webster or the method of Hamilton appears to be a good compromise. Indiana Representative Floyd J. Fithian has recently advocated the use of Hamilton's method for the 1982 Congressional elections. Two mathematicians, M. L. Balinski and H. P. Young, have picked up where Willcox left off in advocating the adoption of the Webster method. As in the past, the decision as to which apportionment method is to be used may be based largely on political considerations. According to the census figures as of January 1, 1981, Indiana stands to gain one seat from New Mexico if either the method of Webster or the method of Hamilton is used in place of Hill's equal proportions. The Hamilton method would also give one of Montana's seats to California.

A forthcoming book, *Fair Representation* by Balinski and Young (to be published by Yale University Press), reviews the history of apportionment for the House of Representatives and presents a well documented and convincing argument for the adoption of Webster's method. More technical presentations and additional references appear in an article by the same authors in the January 1980 *Proceedings of the National Academy of Sciences U.S.A.* and in a recent technical report by W. F. Lucas, issued by the School of Operations Research and Industrial Engineering at Cornell.

The rather bitter and prolonged argument between Willcox and Huntington is documented by numerous commentaries and letters that appeared in *Science* in 1928 and 1929 and again in 1942.

OTHER APPLICATIONS OF APPORTIONMENT METHODS

Fair political representation is just one of many issues in which the problem of equitable apportionment arises. Any operation involving the allocation of people or of indivisible commodities requires some mathematical method of apportionment. A dean or a high school principal may wish to allocate a given number of full-time faculty positions to departments according to some criterion such as the number of courses offered. A company might want to distribute its annual bonus pool (or salary increments) in multiples of one hundred dollars and in such a