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Fair Allocation

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EPA Allocation Problem

The Environmental Protection Agency (EPA) has mandated improvements in the sewage treatment facilities in the cities of Avon, Barport, Claron, and Delmont. Each city could work separately, but $140 million would be saved by all four working together. Some smaller groups of cities also can save money as indicated in the table to the right.

Coalition	Savings
ABCD	140
ABC	108
ABD	96
ACD	84
AB	24
other	0

For cooperation to occur and the savings to be obtained, there must be a signed agreement among the cities stating how the savings is to be allocated among the cities. With no agreement, each city saves nothing ($0 million). Negotiate or arbitrate an allocation.

A Very Simple Problem

Very Simple	
Coalition	**Savings**
ABCD	100
other	0
Player	**Allocation**
A	25
B	25
C	25
D	25

Definition

An allocation is *efficient* if it is impossible to increase the payoff of one player without decreasing the payoff to another player.

Definition

An allocation is *unbiased* if players who are distinguishable only by their names are allocated the same payoff.

A Somewhat Simple Problem

Somewhat Simple	
Coalition	**Savings**
ABCD	150
ABC	150
other	0
Player	**Allocation**
A	50
B	50
C	50
D	0

Definition

An allocation is *subsidy free* if players that never contribute to or detract from gains are allocated zero.

A Not So Simple Problem

	Not So Simple		Very Simple		Somewhat Simple
Coalition	Savings		Savings		Savings
ABCD	250	=	100	+	150
ABC	150	=	0	+	150
Player	Allocation		Allocation		Allocation
A, B, or C	75	=	25	+	50
D	25	=	25	+	0

Definition

An allocation method is *additive* if whenever a problem is the sum or difference of other problems, the allocation for the original problem is the corresponding sum or difference of the allocations of the other problems.

Efficient, Unbiased, Subsidy Free, & Additive Method

Coalition	Savings		S1		S2		S3		S4		S5
ABCD	140	=	24	+	84	+	72	+	84	–	124
ABC	108	=	24	+	0	+	0	+	84	–	0
ABD	96	=	24	+	0	+	72	+	0	–	0
ACD	84	=	0	+	84	+	0	+	0	–	0
AB	24	=	24	+	0	+	0	+	0	–	0
other	0	=	0	+	0	+	0	+	0	–	0
Player	**Alloc**		**A1**		**A2**		**A3**		**A4**		**A5**
A	61	=	12	+	28	+	24	+	28	–	31
B	33	=	12	+	0	+	24	+	28	–	31
C	25	=	0	+	28	+	0	+	28	–	31
D	21	=	0	+	28	+	24	+	0	–	31

Average Marginal Contribution Method

Theorem (Shapley)

The average marginal contribution method is the only allocation method that is efficient, unbiased, subsidy free, and additive.

Coalition	Savings
ABCD	140
ABC	108
ABD	96
ACD	84
AB	24
other	0

Order	C's Marginal Contribution
ABCD	108 – 24 = 84
ABDC	140 – 96 = 44
ACBD	0 – 0 = 0
…	
ADCB	84 – 0 = 84
…	
Average	600 / 4! = 25

Coalition Game References

- L. S. Shapley, *A value for n-person games*, Contributions to the Theory of Games (A. W. Tucker and H. W. Kuhn, eds.), vol. 2, Ann. Math. Stud., no. 28, Princeton University Press, Princeton, NJ, 1953, pp. 307-317.
- Rick Gillman and David Housman, *Models of Conflict and Cooperation*, American Mathematical Society, Providence, RI, 2009, pp.287-334.

Partner Allocation Problem

Five men and women are to enter into arranged marriages. Each person has definite preferences over potential partners as shown in the table. Find the best matching of men to women.

Man	Most to Least Preferred Woman	Woman	Most to Least Preferred Man
Adam	Iris > Helen > Genni > Fran > Jan	Fran	Adam > Ed > Bob > Cary > Doug
Bob	Iris > Fran > Helen > Jan > Genni	Genni	Bob > Ed > Adam > Cary > Doug
Cary	Fran > Iris > Helen > Genni > Jan	Helen	Ed > Bob > Cary > Doug > Adam
Doug	Fran > Genni > Iris > Jan > Helen	Iris	Ed > Cary > Doug > Adam > Bob
Ed	Fran > Genni > Helen > Iris > Jan	Jan	Bob > Adam > Cary > Doug > Ed

Minimize Rank Sum

The matching Fran-Adam, Genni-Doug, Helen-Ed, Iris-Cary, and Jan-Bob has a rank sum of 4 + 4 + 2 + 2 + 3 + 1 + 5 + 1 + 2 + 1 = 25, and this is the minimum possible rank sum.

Man	Most to Least Preferred Woman	Woman	Most to Least Preferred Man
Adam	Iris > Helen > Genni > Fran > Jan	Fran	Adam > Ed > Bob > Cary > Doug
Bob	Iris > Fran > Helen > Jan > Genni	Genni	Bob > Ed > Adam > Cary > Doug
Cary	Fran > Iris > Helen > Genni > Jan	Helen	Ed > Bob > Cary > Doug > Adam
Doug	Fran > Genni > Iris > Jan > Helen	Iris	Ed > Cary > Doug > Adam > Bob
Ed	Fran > Genni > Helen > Iris > Jan	Jan	Bob > Adam > Cary > Doug > Ed

But Adam prefers Genni to Fran, and Genni prefers Adam to Doug--a divorce in the making!

Deferred Acceptance Algorithm

Man	Most to Least Preferred Woman
Adam	Iris > Helen > Genni > Fran > Jan
Bob	Iris > Fran > Helen > Jan > Genni
Cary	Fran > Iris > Helen > Genni > Jan
Doug	Fran > Genni > Iris > Jan > Helen
Ed	Fran > Genni > Helen > Iris > Jan
Woman	**Most to Least Preferred Man**
Fran	Adam > Ed > Bob > Cary > Doug
Genni	Bob > Ed > Adam > Cary > Doug
Helen	Ed > Bob > Cary > Doug > Adam
Iris	Ed > Cary > Doug > Adam > Bob
Jan	Bob > Adam > Cary > Doug > Ed

F	G	H	I	J
CDE			AB	
E			A	
BE	D		AC	
E	D		C	
E	D	AB	C	
E	D	B	C	
E	AD	B	C	
E	A	B	C	
E	A	B	CD	
E	A	B	C	
E	A	B	C	D

Matching Results

Definitions

A matching is *blocked* if there is a man and woman who each prefer each other over their partners in the matching. A matching is *stable* if it is not blocked.

Theorem (Gale & Shapley)

The deferred acceptance algorithm yields a stable matching.

Theorem (Gale & Shapley)

If μ is the deferred acceptance algorithm matching and ν is a stable matching, then no man prefers his partner in ν to his partner in μ, and no woman prefers her partner in μ to her partner in ν.

Application

National Resident Matching Program

Matching References

- David Gale and Lloyd Shapley, College admissions and the stability of marriage, *American Mathematical Monthly*, **69** (1962), 9-15.
- Alvin E. Roth & Marilda A. Oliveira Sotomayer, *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge: Cambridge University Press (1990).

Apportionment Problem

A country has four states with different populations as shown in the table below.

State	Population	Members
1	9,598	
2	5,868	
3	2,664	
4	1,870	
Total	**20,000**	**100**

The Congress consists of 100 members and each state is apportioned proportional to its population an integer number of members. Find the fairest apportionment.

Hamilton's Method

Give to each state the whole number contained in its quota (its population divided by the total population divided by the house size), and then assign remaining seats to states with the largest quota remainders.

State	Population	Quota Pop / 200	Members
1	9,598	47.99	47 + 1 = 48
2	5,868	29.34	29 + 0 = 29
3	2,664	13.32	13 + 0 = 13
4	1,870	9.35	9 + 1 = 10
Total	**20,000**	**100.00**	**100**

Jefferson's Method

Choose an ideal district size. Give each state its whole number of seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State	Population	Pop / 195.7	Members
1	9,598	49.04	49
2	5,868	29.98	29
3	2,664	13.61	13
4	1,870	9.56	9
Total	**20,000**		**100**

Webster's Method

Choose an ideal district size. Give each state its arithmetically rounded number of seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State	Population	Pop / 198	Members
1	9,598	48.47	48
2	5,868	29.63	30
3	2,664	13.45	13
4	1,870	9.44	9
Total	**20,000**		**100**

The methods of Jefferson and Webster, but not Hamilton, are examples of divisor methods: choose an ideal district size and round according to a fixed rule.

Hill’s Method

Choose the apportionment that minimizes the relative difference in average representation between pairs of states.

State	Population	Members	Alt. Members
2	5,868	30	29
4	1,870	9	10
Pairwise Measure of Inequity		$\dfrac{\frac{30}{5868}-\frac{9}{1870}}{\frac{30}{5868}}$	$\dfrac{\frac{10}{1870}-\frac{29}{5868}}{\frac{10}{1870}}$
		≈ 0.0586	≈ 0.0758

This is a divisor method where rounding is with respect to the geometric, rather than arithmetic, mean.

Which Method Is Best?

- “Since the world began there has been but one way of proportioning numbers, namely,
- <insert your favorite method here>
- nor can there be any other method. This process is purely arithmetical . . . If a hundred men were being torn limb from limb, or a thousand babes were being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog.”
- Representative John A. Anderson of Kansas Congressional Record 1882, 12:1179

Fundamental Fairness Properties

Definition

A method satisfies *weak proportionality* if whenever the populations are proportional to an apportionment $\boldsymbol{a}$, then the method selects $\boldsymbol{a}$.

Definition

A method satisfies *completeness* if whenever $\boldsymbol{a}$ is selected for a fixed house size and a sequence of population vectors $\boldsymbol{p}^n$ that converge to the population vector $\boldsymbol{p}$, then $\boldsymbol{a}$ is selected for $\boldsymbol{p}$.

Definition

A method satisfies *symmetry* if permuting the populations results in permuting the apportionments in the same way.

Fairness Properties and Result

Definition

A method satisfies *population monotonicity* if no state that increases its population loses a seat to another state that decreases its population.

Definition

A method satisfies *near quota* if the transfer of a seat from one state to another does not simultaneously take both states closer to their quota.

Theorem (Balinski & Young)

Webster's method is the unique method satisfying weak proportionality, completeness, symmetry, population monotonicity and near quota.

Apportionment References

- Michel L. Balinski and H. Peyton Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, Yale University Press, New Haven, CT, 1982.
- William F. Lucas and David Housman, Apportionment: reflections on the politics of mathematics, *Engineering: Cornell Quarterly*, vol. 16, 1982, pp. 16-22.

Bankruptcy Allocation Problem

A man dies, leaving debts of 100, 200, and 300 to three creditors. Unfortunately, the man's estate is worth less than 600. How should the estate be divided among the creditors?

The Babylonian Talmud stipulates the divisions for three cases, as shown in the table below. What are the guiding principles?

Allocation		Debt			
		100	200	300	
Estate	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	Equal split?
	200	50	75	75	???
	300	50	100	150	Proportional?

Contested Garment Principle

“Two hold a garment; one claims it all, the other claims half. Then the one is awarded ¾ and the other ¼.” (Baba Metzia 2a)

“The lesser claimant concedes half the garment to the greater one. It is only the remaining half that is at issue; this remaining half is therefore divided equally.” (Aumann and Maschler on an eleventh century commentary by Rabbi Shlomo Yitzhaki)

Allocation		Debt	
		1	1/2
Estate	1	1/2 + 1/4 = 3/4	0 + 1/4 = 1/4

More generally, any amount available in the estate above the debt claimed by one person is conceded to the other person; what remains afterward is divided equally.

Contested Garment Principle

More generally, any amount available in the estate above the debt claimed by one person is conceded to the other person; what remains afterward is divided equally.

Allocation		Debt	
		20	50
Estate	10	0 + 5 = 5	0 + 5 = 5
	20	0 + 10 = 10	0 + 10 = 10
	30	0 + 10 = 10	10 + 10 = 20
	40	0 + 10 = 10	20 + 10 = 30
	50	0 + 10 = 10	30 + 10 = 40
	60	10 + 5 = 15	40 + 5 = 45
	60	20 + 0 = 20	50 + 0 = 50

Contested Garment Principle

The first dollars in the estate are divided equally until the smaller claimant obtains half of her claim. The larger claimant then obtains all additional dollars until she is losing the same amount as the smaller claimant. Both claimants equally divide any additional dollars.

Allocation		Debt	
		20	50
Estate	10	0 + 5 = 5	0 + 5 = 5
	20	0 + 10 = 10	0 + 10 = 10
	30	0 + 10 = 10	10 + 10 = 20
	40	0 + 10 = 10	20 + 10 = 30
	50	0 + 10 = 10	30 + 10 = 40
	60	10 + 5 = 15	40 + 5 = 45
	60	20 + 0 = 20	50 + 0 = 50

Contested Garment Principle Consistency

More generally, any amount available in the estate above the debt claimed by one person is conceded to the other person; what remains afterward is divided equally.

Allocation		Debt		
		100	200	300
Estate	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	200	50	75	75
	300	50	100	150

	100	200
150	0 + 50	50 + 50

	100	300
200	0 + 50	100 + 50

	200	300
250	0 + 100	50 + 100

Contested Garment Principle Consistency

More generally, any amount available in the estate above the debt claimed by one person is conceded to the other person; what remains afterward is divided equally.

Allocation		Debt		
		100	200	300
Estate	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	200	50	75	75
	300	50	100	150

	100	200
125	0 + 50	25 + 50

	100	300
125	0 + 50	25 + 50

	200	300
150	0 + 75	0 + 75

Contested Garment Principle Consistency

More generally, any amount available in the estate above the debt claimed by one person is conceded to the other person; what remains afterward is divided equally.

Allocation		Debt		
		100	200	300
Estate	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	200	50	75	75
	300	50	100	150

	$X \geq 100$	$Y \geq 100$
$66\frac{2}{3}$	$0 + 33\frac{1}{3}$	$0 + 33\frac{1}{3}$

Two Questions:

Will there always be an allocation that is consistent with the contested garment principle?

Is an allocation that is consistent with the contested garment principle unique?

Equal Gain/Loss Method

- With an estate of 0, each claimant receives 0.
- Divide each additional dollar equally until a claimant receives half of her claim.
- Divide each additional dollar equally among claimant who have not received half of their claims.
- If all claimants can receive at least half of their claims, then divide losses equally among claimants who have not lost half of their claims.

Allocation		Debt		
		100	200	300
Estate	50	$16\frac{2}{3}$	$16\frac{2}{3}$	$16\frac{2}{3}$
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
	150	50	50	50
	200	50	75	75
	250	50	100	100
	300	50	100	150
	350	50	100	200
	400	50	125	225
	450	50	150	250
	500	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
	550	$83\frac{1}{3}$	$183\frac{1}{3}$	$283\frac{1}{3}$

Bankruptcy Results

Theorem (Aumann & Maschler)

The equal gain/loss method yields allocations that are consistent with the contested garment principle.

Proof. Consider two claimants and verify that the amounts allocated to them in accordance with the equal gain/loss method corresponds to our earlier description of what happens when the contested garment principle is applied.

Theorem (Aumann & Maschler)

The equal gain/loss method yields the only allocations that are consistent with the contested garment principle.

Proof. Suppose for some problem there are two different allocations x and y consistent with the CGP. There must be two claimants i and j satisfying $y_i > x_i$, $y_j < x_j$, and $y_i + y_j \geq x_i + x_j$. If only i and j are involved, CGP awards y_j to j when the estate is $y_i + y_j$, and x_j when the estate is $x_i + x_j$. Since $y_i + y_j \geq x_i + x_j$, the monotonicity of the CGP implies $y_j \geq x_j$, contradicting $y_j < x_j$.

Talmudic Bankruptcy Reference

- Robert J. Aumann and Michael Maschler, Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory*, vol. 36, 1985, pp. 195-213.