An Extremely Simple Fair Division Problem

How to Divide a Chocolate Bar When Different People Value it Differently

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Hard Fair Division Problem

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- The cake is not homogeneous.
- Personal preferences may be complex.
- There may be strategic effects.
- "I divide, you choose" is the oft-cited solution.

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- The chocolate bar is homogeneous.
- Perfectly accurate measuring and cutting devices are available.
- Each person is honest, self-interested, and finds twice as much chocolate twice as good.
- Only the chocolate bar can be allocated.

Simple Fair Division Problem

- Three people have equal shares in a chocolate bar.
- The chocolate bar is homogeneous.
- Perfectly accurate measuring and cutting devices are available.
- Each person is honest, self-interested, and finds twice as much chocolate twice as good.
- People are willing to exchange money as well as receive chocolate.
- Each person attaches a monetary value to the chocolate.

Equal Shares Method

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- In general, if Ann, Ben, and Celine value the chocolate bar at $A > B > C$, then give the chocolate bar to Ann and have her pay $\frac{AB}{A+B+C}$ to Ben and $\frac{AC}{A+B+C}$ to Celine. Each receives a $\frac{A}{A+B+C}$ monetary share.

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Celine envies Ben.

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No person envies another person.

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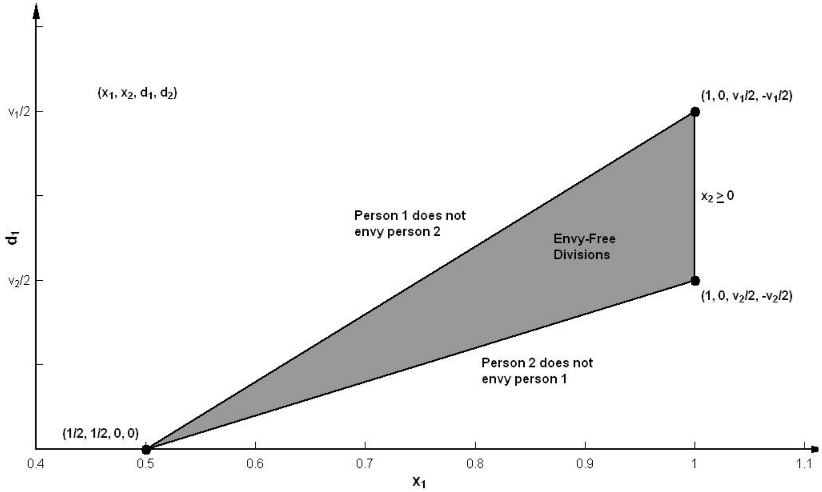
- So, the set of envy-free divisions is the intersection of m^2 half-spaces.

2-Person Envy-Free Divisions



3-Person Extreme Envy-Free Divisions

The extreme vectors of the set of envy-free divisions for a 2-person fair division problem:

	x_1	x_2	d_1	d_2
L_1	1	0	$\frac{1}{2}v_1$	$-\frac{1}{2}v_1$
W_1	1	0	$\frac{1}{2}v_2$	$-\frac{1}{2}v_2$
E	$\frac{1}{2}$	$\frac{1}{2}$	0	0

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E	$\frac{1}{2}$	$\frac{1}{2}$	0	0

The extreme vectors of the set of envy-free divisions for a 3-person fair division problem:

	x_1	x_2	x_3	d_1	d_2	d_3
L_1	1	0	0	$\frac{2}{3}v_1$	$-\frac{1}{3}v_1$	$-\frac{1}{3}v_1$
W_1	1	0	0	$\frac{2}{3}v_2$	$-\frac{1}{3}v_2$	$-\frac{1}{3}v_2$
L_2	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{6}v_2$	$\frac{1}{6}v_2$	$-\frac{1}{3}v_2$
W_2	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{6}v_3$	$\frac{1}{6}v_3$	$-\frac{1}{3}v_3$
E	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0

4-Person Extreme Envy-Free Divisions

The extreme vectors of the set of envy-free divisions for a 4-person fair division problem:

	x_1	x_2	x_3	x_4	d_1	d_2	d_3	d_4
L_1	1	0	0	0	$\frac{3}{4}v_1$	$-\frac{1}{4}v_1$	$-\frac{1}{4}v_1$	$-\frac{1}{4}v_1$
W_1	1	0	0	0	$\frac{3}{4}v_2$	$-\frac{1}{4}v_2$	$-\frac{1}{4}v_2$	$-\frac{1}{4}v_2$
L_2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{4}v_2$	$\frac{1}{4}v_2$	$-\frac{1}{4}v_2$	$-\frac{1}{4}v_2$
W_2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{4}v_3$	$\frac{1}{4}v_3$	$-\frac{1}{4}v_3$	$-\frac{1}{4}v_3$
L_3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{12}v_3$	$\frac{1}{12}v_3$	$\frac{1}{12}v_3$	$-\frac{1}{4}v_3$
W_3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{12}v_4$	$\frac{1}{12}v_4$	$\frac{1}{12}v_4$	$-\frac{1}{4}v_4$
E	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0

m-Person Extreme Envy-Free Divisions

SETH UNRUH'S THEOREM. Suppose there are m persons who can divide a single homogenous object of monetary value v_i to person i, and they are willing to transfer money. The set of envy-free divisions is the simplex with vertices $L_1, L_2, \ldots, L_{m-1}, W_1, W_2, \ldots, W_{m-1}, E$ where

	$x_1 = \cdots = x_k$	$x_{k+1} = \cdots = x_m$	$d_1 = \cdots = d_k$	$d_{k+1} = \cdots = d_m$
L_k	$\frac{1}{k}$	0	$\frac{v_k}{k} - \frac{v_k}{m}$	$-\frac{v_k}{m}$
L_k	$\frac{1}{k}$	0	$\frac{v_{k+1}}{k} - \frac{v_{k+1}}{m}$	$-\frac{v_{k+1}}{m}$
E	$\frac{1}{m}$	$\frac{1}{m}$	0	0

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Proof that the above described divisions are envy-free:

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$$j < l \qquad \text{supposition}$$

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Proof that more of the object is given to those who value the object more:

$$\begin{aligned}
j &< l && \text{supposition} \\
v_j x_j - d_j &\geq v_j x_l - d_l && j \text{ does not envy } l \\
v_l x_l - d_l &\geq v_l x_j - d_j && l \text{ does not envy } j \\
v_j x_j + v_l x_l &\geq v_j x_l + v_l x_j && \text{sum the two inequalities} \\
(v_j - v_l) x_j &\geq (v_j - v_l) x_l && \text{rearrange}
\end{aligned}$$

Envy-Free Divisions as Auctions

~~The extreme vectors of the set of~~ Each envy-free division for an m-person fair division problem can be described in the following manner.

- Think of v_i as a bid person i places on the object.
- Split a portion p of the object equally among the k highest bidders.
- Choose a number between v_k and v_{k+1} to be the winning bid.
- Have each of the k highest bidders pay $\frac{1}{k}p$ of the winning bid.
- Give each person $\frac{1}{m}$ of the money paid.
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x_j &\geq x_l && \text{divide by } (v_j - v_l)
\end{aligned}
$$

Seth Unruh's Proof

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- The set of envy-free divisions is the intersection of the m^2 half-spaces $x_k \geq 0$ and $v_k x_k - d_k \geq v_k x_j - d_j$ for all $k, j = 1, 2, \ldots, m$.

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- L_k, W_k, and E are envy-free divisions for all $k = 1, 2, \ldots, m-1$.

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- L_k, W_k, and E are envy-free divisions for all $k = 1, 2, \ldots, m-1$.
- If (x, d) is an envy-free division, then

$$(x, d) = \sum_{k=1}^{m-1} \alpha_k L_k + \sum_{k=1}^{m-1} \beta_k W_k + \gamma E$$

where

$$\alpha_k = k \frac{(v_{k+1} x_{k+1} - d_{k+1}) - (v_{k+1} x_k - d_k)}{v_k - v_{k+1}}$$

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- These $2m - 1$ divisions are affinely independent in a $2m - 2$ dimensional space.

Future Directions

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