Values for Partition Function Form Games

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Coalition versus partition games and values

Values

- Weighted marginal sum values
- Dispersion values
- Apex values
- Summary values

Properties

- Efficient, symmetric, dummy, and linear
- Monotone
- Dummy independence
Definition (Coalition Game)

Players $N$ and a real-valued function $w$ on $C = \{S : \emptyset \neq S \subseteq N\}$.

Example

<table>
<thead>
<tr>
<th>$S$</th>
<th>$w(S)$</th>
</tr>
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<tbody>
<tr>
<td>ABC</td>
<td>24</td>
</tr>
<tr>
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<tr>
<td>AC</td>
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<tr>
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<td>C</td>
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</tbody>
</table>

Definition (Partition Game)
Players $N$ and a real-valued function $w$ on $\mathcal{E} = \{(S; P) : S \in C_n \text{ and } P \in \mathcal{P}(N \setminus S)\}$.

Example

<table>
<thead>
<tr>
<th>$S; P$</th>
<th>$w(S; P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABC; $\emptyset$</td>
<td>24</td>
</tr>
<tr>
<td>AB; C</td>
<td>18</td>
</tr>
<tr>
<td>AC; B</td>
<td>18</td>
</tr>
<tr>
<td>BC; A</td>
<td>18</td>
</tr>
<tr>
<td>A; B, C</td>
<td>12</td>
</tr>
<tr>
<td>B; A, C</td>
<td>6</td>
</tr>
<tr>
<td>C; B, C</td>
<td>0</td>
</tr>
<tr>
<td>A; BC</td>
<td>0</td>
</tr>
<tr>
<td>B; AC</td>
<td>0</td>
</tr>
<tr>
<td>C; AB</td>
<td>0</td>
</tr>
</tbody>
</table>
For 3-player coalition games:

\[ \phi_A (w) = c(0) \left( w(ABC) - w(BC) \right) + c(1) \left( w(AB) - w(B) \right) + c(1) \left( w(AC) - w(C) \right) + c(2) w(A) \]
Shapley Values
Symmetric Weighted Sum of Marginals

For 3-player coalition games:

\[
\phi_A (w) = c(0) (w(ABC) - w(BC)) + c(1) (w(AB) - w(B)) + c(1) (w(AC) - w(C)) + c(2) w(A)
\]

For 3-player partition games

\[
\phi_A (w) = c(\emptyset; 0) (w(ABC; \emptyset) - w(BC; A)) + c(1; 1) (w(AB; C) - w(B; AC)) + c(1; 0) (w(AB; C) - w(B; A, C)) + c(1; 1) (w(AC; B) - w(C; AB)) + c(1; 0) (w(AC; B) - w(C; A, B)) + c(2) w(A; BC) + c(1; 1) w(A; B, C)
\]
For 3-player coalition games:

\[
\phi_A(w) = \frac{1}{3} (w(ABC) - w(BC)) + \frac{1}{6} (w(AB) - w(B)) + \frac{1}{6} (w(AC) - w(C)) + \frac{1}{3} w(A)
\]
For 3-player coalition games:

\[
\phi_A (w) = \frac{1}{3} (w(ABC) - w(BC)) + \frac{1}{6} (w(AB) - w(B)) + \frac{1}{6} (w(AC) - w(C)) + \frac{1}{3} w(A)
\]

For 3-player partition games

\[
\phi_A (w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A)) + c(1; 1) (w(AB; C) - w(B; AC)) + (\frac{1}{6} - c(1; 1)) (w(AB; C) - w(B; A, C)) + c(1; 1) (w(AC; B) - w(C; AB)) + (\frac{1}{6} - c(1; 1)) (w(AC; B) - w(C; A, B)) + 2c(1; 1) w(A; BC) + (\frac{1}{3} - 2c(1; 1)) w(A; B, C)
\]
Shapley Values

Efficient and Symmetric Weighted Sum of Marginals

For 3-player coalition games:

\[
\phi_A (w) = \frac{1}{3} (w(ABC) - w(BC)) \\
+ \frac{1}{6} (w(AB) - w(B)) \\
+ \frac{1}{6} (w(AC) - w(C)) \\
+ \frac{1}{3} w(A)
\]

For 3-player partition games

\[
\phi_A (w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A)) \\
+ c(1; 1) (w(AB; C) - w(B; AC)) \\
+ (\frac{1}{6} - c(1; 1)) (w(AB; C) - w(B; A, C)) \\
+ c(1; 1) (w(AC; B) - w(C; AB)) \\
+ (\frac{1}{6} - c(1; 1)) (w(AC; B) - w(C; A, B)) \\
+ 2c(1; 1) w(A; BC) \\
+ (\frac{1}{3} - 2c(1; 1)) w(A; B, C)
\]

\[c(1; 1) = \phi_A (w) \text{ if } w(ABC; \emptyset) = w(BC; A) = w(AB; C) = 1.\]
Shapley Values
For 5-player partition games

\[
\phi_A (w) = \frac{1}{5} (w (ABCDE; \emptyset) - w (BCDE; A)) + c(1; 1) (w (ABCD; E) - w (BCD; AE)) \\
\ldots + c(2; 2) (w (ABC; DE) - w (BC; ADE)) \\
\ldots + c(1, 1; 1) (w (ABC; D, E) - w (BC; AD, E)) \\
\ldots + c(3; 3) (w (AB; CDE) - w (B; ACDE)) \\
\ldots + c(2, 1; 2) (w (AB; CD, E) - w (B; ACD, E)) \\
\ldots + c(2, 1; 1) (w (AB; CD, E) - w (B; CD, AE)) \\
\ldots + c(1, 1, 1; 1) (w (AB; C, D, E) - w (B; AC, D, E)) \\
\ldots
\]

In 3, 4, and 5 player games, there are 1, 3, and 7 parameters.
For coalition games:

\[ SH_i(w) = \sum_{S \in \mathcal{C}_n : i \in S} \frac{(|S| - 1)! (n - |S|)!}{n!} (w(S) - w(S \setminus \{i\})) \]
For coalition games:

$$SH_i(w) = \sum_{S \in \mathcal{C}_n:i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} (w(S) - w(S \setminus \{i\}))$$

For partition games

$$\phi^c_i(w) = \sum_{(S;P)} \sum_{R \in P \cup \{\emptyset\}} c_n(\|P\|;|R|) (w(S;P) - w(S \setminus \{i\};P[i,R]))$$

$$+ \sum_{P \in \mathcal{P}(N \setminus \{i\})} c_n(\|P\|) w(\{i\};P)$$

where the first sum is over embedded coalitions $(S;P)$ satisfying $i \in S$ and $|S| \geq 2$, $c_n(p;r) \in \mathbb{R}$ for all $p \in \mathcal{P}^*(n-2)$ and $r \in p$ are parameters, and the other coefficients are defined in terms of the parameters.
Axiomatic Characterizations

Efficient \[ \sum_{i \in N} \phi_i(w) = w(N; \emptyset). \]

Symmetric Relabeling players relabels the payoffs.

Dummy If a player cannot unilaterally change the worth of an (embedded) coalition, then that player receives a zero payoff.

Linear \[ \phi(a_1 w^1 + a_2 w^2) = a_1 \phi(w^1) + a_2 \phi(w^2). \]
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Theorem (Shapley, 1953)

The Shapley coalition value is the unique efficient, symmetric, dummy, and linear coalition value.
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**Theorem (Shapley, 1953)**

The Shapley coalition value is the unique efficient, symmetric, dummy, and linear coalition value.

**Theorem (McCaulley, 1990; Merki, 1991; Theoharidis, 1993)**

The Shapley partition values are the only efficient, symmetric, dummy, and linear partition values.
\[ S \quad w(S) = w^1(S) + w^2(S) + w^3(S) + w^4(S) - w^5(S) \]

<table>
<thead>
<tr>
<th></th>
<th>w^1(S)</th>
<th>w^2(S)</th>
<th>w^3(S)</th>
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<tbody>
<tr>
<td>ABC</td>
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<td>12</td>
<td>18</td>
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</tr>
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<td>AB</td>
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<td>12</td>
<td>0</td>
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<td>6</td>
</tr>
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<td>C</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>

\[ i \quad \phi_i(w) = \phi_i(w^1) + \phi_i(w^2) + \phi_i(w^3) + \phi_i(w^4) - \phi_i(w^5) \]

<table>
<thead>
<tr>
<th></th>
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<th>\phi_i(w^2)</th>
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<th>\phi_i(w^4)</th>
<th>\phi_i(w^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>12</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>0</td>
<td>9</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>0</td>
<td>9</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
Efficient, Symmetric, Dummy, and Linear Partition Values

\[ S; P \quad w(S; P) = w^1(S; P) + w^2(S; P) + w^3(S; P) + w^4(S; P) - w^5(S; P) \]

<table>
<thead>
<tr>
<th></th>
<th>( S; P )</th>
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<tbody>
<tr>
<td>( A; B; C )</td>
<td>24</td>
<td>6</td>
</tr>
<tr>
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<td>18</td>
<td>6</td>
</tr>
<tr>
<td>( C; B; A )</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( A; BC )</td>
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<td>0</td>
</tr>
<tr>
<td>( B; AC )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( C; AB )</td>
<td>0</td>
<td>0</td>
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\[
i \quad \phi_i(w) = \phi_i(w^1) + \phi_i(w^2) + \phi_i(w^3) + \phi_i(w^4) - \phi_i(w^5)\]

<table>
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<tr>
<th></th>
<th>( \phi_i(w) )</th>
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<tr>
<td>( A )</td>
<td>11 - 3( \alpha )</td>
</tr>
<tr>
<td>( B )</td>
<td>8</td>
</tr>
<tr>
<td>( C )</td>
<td>5 + 3( \alpha )</td>
</tr>
</tbody>
</table>
The coalition game value characterization holds on any positive convex cone of games containing the unanimity games $u^T(S) = 1$ if $T \subseteq S$.

Myerson (1977) proved there is a unique additive, symmetric, and carrier value. Carrier is redundant on partition competitive games.
The coalition game value characterization holds on any positive convex cone of games containing the unanimity games \( u^T(S) = 1 \) if \( T \subseteq S \).

With 4-player unanimity games, 4 parameters appear, but the formula has only 3.
The coalition game value characterization holds on any positive convex cone of games containing the unanimity games $u^T(S) = 1$ if $T \subseteq S$.

With 4-player unanimity games, 4 parameters appear, but the formula has only 3.

\[
\begin{align*}
E & \quad u^A;BD,C(E) + u^A;B,CD(E) = u^A;B,C,D(E) + d^A;D,B,C_E(E) \\
AB;CD & \quad 0 \quad 1 \quad 0 \quad 1 \\
AC;BD & \quad 1 \quad 0 \quad 0 \quad 1 \\
A;B,C,D & \quad 1 \quad 1 \quad 1 \quad 1 \\
A;B,CD & \quad 0 \quad 1 \quad 0 \quad 1 \\
A;BD,C & \quad 1 \quad 0 \quad 0 \quad 1 \\
\end{align*}
\]

\[
\begin{align*}
i & \quad \phi_i(u^A;BD,C) + \phi_i(u^A;B,CD) = \phi_i(u^A;B,C,D) + \phi_i(d^A;D,B,C) \\
D & \quad b \quad b \quad c \quad 0
\end{align*}
\]
The coalition game value characterization holds on any positive convex cone of games containing the unanimity games $u^T(S) = 1$ if $T \subseteq S$.

With 4-player unanimity games, 4 parameters appear, but the formula has only 3.

$$E u^{A;BD,C}(E) + u^{A;B,CD}(E) = u^{A;B,C,D}(E) + d^{AD;B,C}_D(E)$$

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<tr>
<th></th>
<th>AB;CD</th>
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<th>A;B,C,D</th>
<th>A;B,CD</th>
<th>A;BD,C</th>
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<td>u^{A;BD,C}</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>u^{A;B,CD}</td>
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<td>0</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>

$$i \phi_i(u^{A;BD,C}) + \phi_i(u^{A;B,CD}) = \phi_i(u^{A;B,C,D}) + \phi_i(d^{AD;B,C}_D)$$

| D | b | b | c | 0 |

The partition game value characterization holds on any convex cone of games containing the dummy games

$$d^T_i(S;P) = 1 \text{ if } (T \setminus \{i\}; Q[i,R]) \preceq (S;P) \text{ for some } R \in Q \cup \{\emptyset\}.$$
The coalition game value characterization holds on any positive convex cone of games containing the unanimity games \( u^T(S) = 1 \) if \( T \subseteq S \).

With 4-player unanimity games, 4 parameters appear, but the formula has only 3.

\[
\begin{align*}
E & \quad u^{A;BD,C}(E) + u^{A;B,CD}(E) = u^{A;B,C,D}(E) + d^{AD;B,C}_D(E) \\
AB;CD & \quad 0 \quad 1 \quad 0 \quad 1 \\
AC;BD & \quad 1 \quad 0 \quad 0 \quad 1 \\
A;B,C,D & \quad 1 \quad 1 \quad 1 \quad 1 \\
A;B,CD & \quad 0 \quad 1 \quad 0 \quad 1 \\
A;BD,C & \quad 1 \quad 0 \quad 0 \quad 1 
\end{align*}
\]

\[
\begin{align*}
\sum_i \phi_i(u^{A;BD,C}) + \phi_i(u^{A;B,CD}) = \phi_i(u^{A;B,C,D}) + \phi_i(d^{AD;B,C}_D) \\
D & \quad b \quad b \quad c \quad 0
\end{align*}
\]

The partition game value characterization holds on any convex cone of games containing the dummy games

\[
d^{T;Q}_i(S; P) = 1 \text{ if } (T \setminus \{i\}; Q[i, R]) \lesssim (S; P) \text{ for some } R \in Q \cup \{\emptyset\}.
\]

Myerson (1977) proved there is a unique additive, symmetric, and carrier value. Carrier is redundant on partition competitive games.
Dispersion Process

For coalition games:

\[ \text{ABC} = \frac{1}{3} \]

\[ \text{BC} = \frac{1}{2} \]

\[ \text{C} = 1 \]

\[ w(\text{ABC}) \]
\[ w(\text{BC}) \]
\[ w(\text{BC}) \]
\[ w(\text{C}) \]
\[ w(\text{C}) \]

\[ \text{to A to B to C} \]

For partition games:

\[ \text{ABC}; \emptyset = \frac{1}{3} \]

\[ \text{BC}; \emptyset = \frac{1}{2} \]

\[ \text{C}; \emptyset = \frac{1}{2} \]

\[ w(\text{ABC}; \emptyset) \]
\[ w(\text{BC}; \emptyset) \]
\[ w(\text{BC}; \emptyset) \]
\[ w(\text{C}; \emptyset) \]
\[ w(\text{C}; \emptyset) \]

\[ \text{to A to B to C} \]

Where \[ a(1; 0) + a(1; 0) = 1. \]
Dispersion Process

For coalition games:

\[ ABC \]

For partition games:

\[ ABC; \emptyset =\frac{1}{3} \]

\[ BC; A = \frac{1}{2} a(1;0) \]

\[ C; AB = \emptyset \]

\[ w(ABC; \emptyset) \]

\[ w(BC; A) \]

\[ w(BC; A) \]

\[ w(C; AB) \]

\[ w(C; AB) \]

\[ \text{to } A \]
Dispersion Process

For coalition games:

\[ \begin{align*}
ABC & \rightarrow BC \\
& \frac{1}{3}
\end{align*} \]
Dispersion Process

For coalition games:

\[
\begin{align*}
A B C & \quad \Rightarrow \quad 1/3 \\
B C & \quad \Rightarrow \quad 1/2 \\
C & 
\end{align*}
\]
Dispersion Process

For coalition games:

\[ \begin{align*}
ABC & \rightarrow 1/3 \\
BC & \rightarrow 1/2 \\
C & \rightarrow 1 \\
\emptyset &
\end{align*} \]
Dispersion Process

For coalition games:

\[
\begin{align*}
ABC & \rightarrow_{1/3} \emptyset \\
BC & \rightarrow_{1/2} \emptyset \\
C & \rightarrow_1 \emptyset
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>(w(ABC) - w(BC)) to A</th>
<th>(w(BC) - w(C)) to B</th>
<th>(w(C)) to C</th>
</tr>
</thead>
</table>

where \(a(1;1) + a(1;0) = 1\).
Dispersion Process

For coalition games:
\[
\begin{align*}
ABC & \xrightarrow{1/3} BC \xrightarrow{1/2} C \xrightarrow{1} \emptyset \\
\end{align*}
\]

\[
\begin{align*}
w(ABC) - w(BC) & \quad w(BC) - w(C) & \quad w(C) \\
to A & \quad to B & \quad to C
\end{align*}
\]

For partition games:
\[
\begin{align*}
ABC; \emptyset & \xrightarrow{1/3} BC; A \xrightarrow{1/2} C; AB \xrightarrow{1} \emptyset \\
\end{align*}
\]

\[
\begin{align*}
w(ABC; \emptyset) - w(BC; A) & \quad w(BC; A) - w(C; AB) & \quad w(C; AB) \\
to A & \quad to B & \quad to C
\end{align*}
\]

\[
\begin{align*}
ABC; \emptyset & \xrightarrow{1/3} BC; A \xrightarrow{1/2} C; A, B \xrightarrow{1} \emptyset \\
\end{align*}
\]

\[
\begin{align*}
w(ABC; \emptyset) - w(BC; A) & \quad w(BC; A) - w(C; A, B) & \quad w(C; A, B) \\
to A & \quad to B & \quad to C
\end{align*}
\]

where \(a(1; 1) + a(1; 0) = 1\).
\[ \phi_a^A(w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A)) + \frac{1}{3} \frac{1}{2} a(1; 1) (w(AB; C) - w(B; AC)) + \frac{1}{3} \frac{1}{2} a(1; 0) (w(AB; C) - w(B; A, C)) + \frac{1}{3} \frac{1}{2} a(1; 1) (w(AC; B) - w(C; AB)) + \frac{1}{3} \frac{1}{2} a(1; 0) (w(AC; B) - w(C; A, B)) + 2 \frac{1}{3} \frac{1}{2} a(1; 1) w(A; BC) + 2 \frac{1}{3} \frac{1}{2} a(1; 0) w(A; B, C) \]
Dispersion and Apex Values

\[
\phi^a_A(w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A))
+ \frac{1}{3} \frac{1}{2} a(1; 1) (w(AB; C) - w(B; AC))
+ \frac{1}{3} \frac{1}{2} a(1; 0) (w(AB; C) - w(B; A, C))
+ \frac{1}{3} \frac{1}{2} a(1; 1) (w(AC; B) - w(C; AB))
+ \frac{1}{3} \frac{1}{2} a(1; 0) (w(AC; B) - w(C; A, B))
+ 2 \frac{1}{3} \frac{1}{2} a(1; 1) w(A; BC)
+ 2 \frac{1}{3} \frac{1}{2} a(1; 0) w(A; B, C)
\]

\[
\phi^{12}_A(w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A))
+ \frac{1}{3} \frac{1}{6} (w(AB; C) - w(B; AC))
+ \frac{1}{3} \frac{1}{6} (w(AC; B) - w(C; AB))
+ \frac{1}{3} w(A; BC)
\]

\[
\phi^{1,2}_A(w) = \frac{1}{3} (w(ABC; \emptyset) - w(BC; A))
+ \frac{1}{3} \frac{1}{6} (w(AB; C) - w(B; A, C))
+ \frac{1}{3} \frac{1}{6} (w(AC; B) - w(C; A, B))
+ \frac{1}{3} w(A; B, C)
\]
Main Characterization

Theorem (Housman, 1996)

Suppose $\phi$ is a value on a convex cone $\Gamma$ containing dummy games. The following statements are equivalent:

1. $\phi$ is a Shapley value.
2. $\phi$ is a dispersion value.
3. $\phi$ is an affine combination of apex values.
4. $\phi$ is efficient, symmetric, dummy, and linear on $\Gamma$. 
Second Characterization

Monotone \( \phi_i(v) \leq \phi_i(w) \) whenever \( i \in N \) and \( v, w \in \Gamma_n \) satisfy
\( v(S; P) \leq w(S; P) \) if \( i \in S \) and \( v(S; P) \geq w(S; P) \) if \( i \notin S \).
Theorem (Housman, 1996)

Suppose $\phi$ is a value on a convex cone $\Gamma$ containing dummy games and having a nonempty interior. The following statements are equivalent:

1. $\phi$ is a Shapley value with nonnegative coefficients.
2. $\phi$ is a dispersion value with nonnegative parameters.
3. $\phi$ is a convex combination of apex values.
4. $\phi$ is efficient, symmetric, dummy, linear, and monotone on $\Gamma$. 

Second Characterization

Monotone $\phi_i(v) \leq \phi_i(w)$ whenever $i \in N$ and $v, w \in \Gamma_n$ satisfy $v(S;P) \leq w(S;P)$ if $i \in S$ and $v(S;P) \geq w(S;P)$ if $i \notin S$. 

Different Numbers of Players

$S; P \ w(S; P)$

$ABC; \emptyset \ 1$

$AB; C \ 1$

$AC; B \ 1$

$A; B, C \ 1$

$i \ \phi_i(w)$

$A \ 1 - 2\alpha$

$B \ \alpha$

$C \ \alpha$

$\alpha = c_3(1; 1)$
### Different Numbers of Players

<table>
<thead>
<tr>
<th>S; P</th>
<th>( w(S; P) )</th>
<th>S; P</th>
<th>( w^D(S; P) )</th>
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<tbody>
<tr>
<td>ABC; ( \emptyset )</td>
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<td>ABCD; ( \emptyset )</td>
<td>1</td>
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<tr>
<td></td>
<td></td>
<td>ABC; D</td>
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<td>AB; C</td>
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<td>ABD; C</td>
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<td></td>
<td></td>
<td>AB; C, D</td>
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</tr>
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<td></td>
<td>AB; CD</td>
<td>1</td>
</tr>
<tr>
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<td></td>
<td>AC; BD</td>
<td>1</td>
</tr>
<tr>
<td>A; B, C</td>
<td>1</td>
<td>AD; B, C</td>
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<td>A; B, C, D</td>
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<td>A; BD, C</td>
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<table>
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<tr>
<th>( i )</th>
<th>( \phi_i(w) )</th>
<th>( i )</th>
<th>( \phi_i(w^D) )</th>
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<tbody>
<tr>
<td>A</td>
<td>( 1 - 2\alpha )</td>
<td>A</td>
<td>( 1 - 2\beta )</td>
</tr>
<tr>
<td>B</td>
<td>( \alpha )</td>
<td>B</td>
<td>( \beta )</td>
</tr>
<tr>
<td>C</td>
<td>( \alpha )</td>
<td>C</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( \alpha = c_3(1; 1) )</td>
<td>( \alpha )</td>
<td>D</td>
<td>0</td>
</tr>
</tbody>
</table>
Different Numbers of Players

\[
\begin{array}{ccc}
S; P & w(S; P) & S; P & w^D(S; P) \\
ABC; \emptyset & 1 & ABCD; \emptyset & 1 \\
 & ABC; D & 1 \\
AB; C & 1 & ABD; C & 1 \\
 & AB; C, D & 1 \\
 & AB; CD & 1 \\
AC; B & 1 & ACD; B & 1 \\
 & AC; B, D & 1 \\
 & AC; BD & 1 \\
A; B, C & 1 & AD; B, C & 1 \\
 & A; B, C, D & 1 \\
 & A; B, CD & 1 \\
 & A; BD, C & 1 \\
\end{array}
\]

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters \(c_n\) and \(c_{n+1}\).

\[
\begin{array}{ccc}
i & \phi_i(w) & i & \phi_i(w^D) \\
A & 1 - 2\alpha & A & 1 - 2\beta \\
B & \alpha & B & \beta \\
C & \alpha & C & \beta \\
\alpha = c_3(1; 1) & & D & 0 \\
\end{array}
\]
Different Numbers of Players

<table>
<thead>
<tr>
<th>S; P</th>
<th>w(S; P)</th>
<th>S; P</th>
<th>w^D(S; P)</th>
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<td>AB; C</td>
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<td>AB; C; D</td>
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<tr>
<td>AC; B</td>
<td>1</td>
<td>AC; B; D</td>
<td>1</td>
</tr>
<tr>
<td>A; B; C</td>
<td>1</td>
<td>AD; B; C</td>
<td>1</td>
</tr>
<tr>
<td>A; B; C; D</td>
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<td>A; B; C; D</td>
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</tbody>
</table>

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters c_n and c_{n+1}.

The dummy extension of the partition game (N, w) is the partition game (N ∪ {d}, w^d) defined by w^d(S; P) = w(S \ {d}; {R \ {d} : R ∈ P}) \ {∅} for all embedded coalitions (S; P) of N ∪ {d}.

\[
\begin{align*}
\phi_i(w) & \\
A & 1 - 2\alpha \\
B & \alpha \\
C & \alpha \\
\alpha = c_3(1; 1) & \\
\end{align*}
\]

\[
\begin{align*}
\phi_i(w^D) & \\
A & 1 - 2\beta \\
B & \beta \\
C & \beta \\
D & 0 \\
\end{align*}
\]
Different Numbers of Players

$S; P \quad w(S; P) \quad S; P \quad w^D(S; P)$

$ABC; \emptyset \quad 1 \quad ABCD; \emptyset \quad 1$
$ABC; D \quad 1$
$AB; C \quad 1 \quad ABD; C \quad 1$
$AB; C, D \quad 1$
$AB; CD \quad 1$
$AC; B \quad 1 \quad ACD; B \quad 1$
$AC; B, D \quad 1$
$AC; BD \quad 1$
$A; B, C \quad 1 \quad AD; B, C \quad 1$
$A; B, C, D \quad 1$
$A; B, CD \quad 1$
$A; BD, C \quad 1$

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters $c_n$ and $c_{n+1}$.

The dummy extension of the partition game $(N, w)$ is the partition game $(N \cup \{d\}, w^d)$ defined by $w^d(S; P) = w(S \setminus \{d\}; \{R \setminus \{d\} : R \in P\} \setminus \{\emptyset\})$ for all embedded coalitions $(S; P)$ of $N \cup \{d\}$.

$\begin{array}{c|c}
\alpha & 1 - 2\alpha \\
A & 1 - 2\beta \\
B & \beta \\
C & \beta \\
\alpha = c_3(1; 1) & D & 0
\end{array}$

Dummy Independent $\forall w, w^d \in \Gamma$ and $\forall i \in N$, $\phi_i(w^d) = \phi_i(w)$. 
Summary Values

Given a partition game $w$, define a summary coalition game $w^b$ by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|)w(S; P)$$

where

$$b(\emptyset) = 1.$$
Summary Values

Given a partition game $w$, define a summary coalition game $w^b$ by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|)w(S; P)$$

where

$$b(\emptyset) = 1.$$ 

Define the summary value $\phi^b$ by

$$\phi^b(w) = Sh(w^b).$$
Summary Values

Given a partition game $w$, define a summary coalition game $w^b$ by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|)w(S; P)$$

where

$$b(\emptyset) = 1.$$ 

Define the summary value $\phi^b$ by

$$\phi^b(w) = S_H(w^b).$$

For $n \geq 4$, there are Shapley values that are not summary values, and there are summary values that are not Shapley values.
Summary Values

Given a partition game \( w \), define a summary coalition game \( w^b \) by

\[
w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|)w(S; P)
\]

where

\[
b(\emptyset) = 1.
\]

Define the summary value \( \phi^b \) by

\[
\phi^b(w) = S_H(w^b).
\]

For \( n \geq 4 \), there are Shapley values that are not summary values, and there are summary values that are not Shapley values.

**Theorem**

A summary value \( \phi^b \) is a Shapley value on \( n \)-player games if and only if

\[
b_n(p) = \sum_{r \in p} b_n(p \setminus \{r\} \cup \{r + 1\})
\]

for all \( p \in \mathcal{P}^*(n - 2) \).
Third Characterization

Given a partition game $w$, define a summary coalition game $w^b$ by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|) w(S; P)$$

where

$$b(\emptyset) = 1.$$

Define the summary value $\phi^b$ by

$$\phi^b(w) = S_H(w^b).$$

Theorem

Suppose $\phi$ is a value on a convex cone $\Gamma$ containing dummy games, having a nonempty interior, and closed with respect to dummy extension. A value $\phi$ is efficient, symmetric, linear, monotone, and dummy independent on $\Gamma$ if and only if $\phi$ is a summary value with nonnegative parameters that are identical for all $n$ and satisfy

$$b(p) = \sum_{r \in p} b(p \setminus \{r\} \cup \{r + 1\})$$

for all natural number partitions $p$. 
References