Values for Partition Function Form Games

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Preview

- Coalition versus partition games and values
- Values
  - Weighted marginal sum values
  - Dispersion values
  - Apex values
  - Summary values
- Properties
  - Efficient, symmetric, dummy, and linear
  - Monotone
  - Dummy independence

Games

Definition (Coalition Game)

Players N and a real-valued function w on $\mathcal{C} = \{S : \emptyset \neq S \subseteq N\}$.

Example

S	$w(S)$
ABC	24
AB	18
AC	18
BC	18
A	12
B	0
C	0

Games

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B	0
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Definition (Partition Game)

Players N and a real-valued function w on $\mathcal{E} = \{(S; P) : S \in \mathcal{C}_n \text{ and } P \in \mathcal{P}(N \backslash S)\}$.

Example

$S; P$	$w(S; P)$
$ABC; \emptyset$	24
$AB; C$	18
$AC; B$	18
$BC; A$	18
$A; B, C$	12
$B; A, C$	6
$C; B, C$	0
$A; BC$	0
$B; AC$	0
$C; AB$	0

Shapley Values

Symmetric Weighted Sum of Marginals

For 3-player coalition games:

$$\begin{aligned}\phi_A(w) &= c(0)\quad (w(ABC) - w(BC)) \\ &+ c(1)\quad (w(AB) - w(B)) \\ &+ c(1)\quad (w(AC) - w(C)) \\ &+ c(2)\quad w(A)\end{aligned}$$

For 3-player coalition games:

$$
\begin{aligned}
\phi_A(w) &= c(0) && (w(ABC) - w(BC)) \\
&+ c(1) && (w(AB) - w(B)) \\
&+ c(1) && (w(AC) - w(C)) \\
&+ c(2) && w(A)
\end{aligned}
$$

For 3-player partition games

$$
\begin{aligned}
\phi_A(w) &= c(\emptyset;0) && (w(ABC;\emptyset) - w(BC;A)) \\
&+ c(1;1) && (w(AB;C) - w(B;AC)) \\
&+ c(1;0) && (w(AB;C) - w(B;A,C)) \\
&+ c(1;1) && (w(AC;B) - w(C;AB)) \\
&+ c(1;0) && (w(AC;B) - w(C;A,B)) \\
&+ c(2) && w(A;BC) \\
&+ c(1,1) && w(A;B,C)
\end{aligned}
$$

For 3-player coalition games:

$$\begin{aligned}\phi_A(w) &= \frac{1}{3} && (w(ABC) - w(BC)) \\ &+ \frac{1}{6} && (w(AB) - w(B)) \\ &+ \frac{1}{6} && (w(AC) - w(C)) \\ &+ \frac{1}{3} && w(A)\end{aligned}$$

Shapley Values

Efficient and Symmetric Weighted Sum of Marginals

For 3-player coalition games:

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For 3-player partition games

$$\begin{aligned} \phi_A(w) &= \tfrac{1}{3} && (w(ABC;\emptyset) - w(BC;A)) \\ &+ c(1;1) && (w(AB;C) - w(B;AC)) \\ &+ (\tfrac{1}{6} - c(1;1)) && (w(AB;C) - w(B;A,C)) \\ &+ c(1;1) && (w(AC;B) - w(C;AB)) \\ &+ (\tfrac{1}{6} - c(1;1)) && (w(AC;B) - w(C;A,B)) \\ &+ 2c(1;1) && w(A;BC) \\ &+ (\tfrac{1}{3} - 2c(1;1)) && w(A;B,C) \end{aligned}$$

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$$\begin{aligned}\phi_A(w) &= \tfrac{1}{3}\,(w(ABC) - w(BC)) \\ &+ \tfrac{1}{6}\,(w(AB) - w(B)) \\ &+ \tfrac{1}{6}\,(w(AC) - w(C)) \\ &+ \tfrac{1}{3}\,w(A)\end{aligned}$$

For 3-player partition games

$$\begin{aligned}\phi_A(w) &= \tfrac{1}{3}\,(w(ABC;\emptyset) - w(BC;A)) \\ &+ c(1;1)\,(w(AB;C) - w(B;AC)) \\ &+ (\tfrac{1}{6} - c(1;1))\,(w(AB;C) - w(B;A,C)) \\ &+ c(1;1)\,(w(AC;B) - w(C;AB)) \\ &+ (\tfrac{1}{6} - c(1;1))\,(w(AC;B) - w(C;A,B)) \\ &+ 2c(1;1)\,w(A;BC) \\ &+ (\tfrac{1}{3} - 2c(1;1))\,w(A;B,C)\end{aligned}$$

$c(1;1) = \phi_A(w)$ if $w(ABC;\emptyset) = w(BC;A) = w(AB;C) = 1$.

Shapley Values

For 5-player partition games

$$
\begin{aligned}
\phi_A(w) &= & \tfrac{1}{5} & \left(w\left(ABCDE;\emptyset\right) - w\left(BCDE;A\right)\right) \\
&+ & c(1;1) & \left(w\left(ABCD;E\right) - w\left(BCD;AE\right)\right) \\
&\cdots \\
&+ & c(2;2) & \left(w\left(ABC;DE\right) - w\left(BC;ADE\right)\right) \\
&\cdots \\
&+ & c(1,1;1) & \left(w\left(ABC;D,E\right) - w(BC;AD,E)\right) \\
&\cdots \\
&+ & c(3;3) & \left(w\left(AB;CDE\right) - w(B;ACDE)\right) \\
&\cdots \\
&+ & c(2,1;2) & \left(w\left(AB;CD,E\right) - w(B;ACD,E)\right) \\
&\cdots \\
&+ & c(2,1;1) & \left(w\left(AB;CD,E\right) - w(B;CD,AE)\right) \\
&\cdots \\
&+ & c(1,1,1;1) & \left(w\left(AB;C,D,E\right) - w(B;AC,D,E)\right) \\
&\cdots
\end{aligned}
$$

In 3, 4, and 5 player games, there are 1, 3, and 7 parameters.

Shapley Values

Efficient and Symmetric Weighted Sum of Marginals

For coalition games:

$$\mathrm{SH}_i\left(w\right) = \sum_{S \in \mathcal{C}_n : i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}\left(w\left(S\right) - w\left(S \setminus \left\{i\right\}\right)\right)$$

For coalition games:

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For partition games

$$\begin{aligned} \phi_i^c(w) &= \sum_{(S;P)} \sum_{R \in P \cup \{\varnothing\}} c_n(\|P\|; |R|) \left(w(S;P) - w(S \setminus \{i\}; P[i,R]) \right) \\ &\quad + \sum_{P \in \mathcal{P}(N \setminus \{i\})} c_n(\|P\|) \, w(\{i\}; P) \end{aligned}$$

where the first sum is over embedded coalitions $(S; P)$ satisfying $i \in S$ and $|S| \geq 2$, $c_n(p; r) \in \mathbb{R}$ for all $p \in \mathcal{P}^*(n-2)$ and $r \in p$ are parameters, and the other coefficients are defined in terms of the parameters.

Axiomatic Characterizations

Efficient $\sum_{i \in N} \phi_i(w) = w(N; \emptyset)$.

Symmetric Relabeling players relabels the payoffs.

Dummy If a player cannot unilaterally change the worth of an (embedded) coalition, then that player receives a zero payoff.

Linear $\phi(a_1 w^1 + a_2 w^2) = a_1 \phi(w^1) + a_2 \phi(w^2)$.

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Theorem (Shapley, 1953)

The Shapley coalition value is the unique efficient, symmetric, dummy, and linear coalition value.

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Theorem (McCaulley, 1990; Merki, 1991; Theoharidis, 1993)

The Shapley partition values are the only efficient, symmetric, dummy, and linear partition values.

Efficient, Symmetric, Dummy, and Linear Coalition Value

S	$w(S)$	$=$	$w^1(S)$	$+$	$w^2(S)$	$+$	$w^3(S)$	$+$	$w^4(S)$	$-$	$w^5(S)$
ABC	24		12		18		6		6		18
AB	18		12		0		0		6		0
AC	18		12		0		6		0		0
BC	18		0		18		0		0		0
A	12		12		0		0		0		0
B	0		0		0		0		0		0
C	0		0		0		0		0		0

i	$\phi_i(w)$	$=$	$\phi_i(w^1)$	$+$	$\phi_i(w^2)$	$+$	$\phi_i(w^3)$	$+$	$\phi_i(w^4)$	$-$	$\phi_i(w^5)$
A	12		12		0		3		3		6
B	6		0		9		0		3		6
C	6		0		9		3		0		6

Efficient, Symmetric, Dummy, and Linear Partition Values

$S; P$	$w(S;P)$	=	$w^1(S;P)$	+	$w^2(S;P)$	+	$w^3(S;P)$	+	$w^4(S;P)$	−	$w^5(S;P)$
$ABC; \emptyset$	24		6		12		12		6		12
$AB; C$	18		6		12		0		0		0
$AC; B$	18		0		12		0		6		0
$BC; A$	18		6		0		12		0		0
$A; B, C$	12		0		12		0		0		0
$B; A, C$	6		6		0		0		0		0
$C; B, C$	0		0		0		0		0		0
$A; BC$	0		0		0		0		0		0
$B; AC$	0		0		0		0		0		0
$C; AB$	0		0		0		0		0		0

i	$\phi_i(w)$	=	$\phi_i(w^1)$	+	$\phi_i(w^2)$	+	$\phi_i(w^3)$	+	$\phi_i(w^4)$	−	$\phi_i(w^5)$
A	$11 - 3\alpha$		α		$12 - 4\alpha$		0		3		4
B	8		$6 - 2\alpha$		2α		6		0		4
C	$5 + 3\alpha$		α		2α		6		3		4

Game Spaces for Axiomatic Characterizations

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E	$\mathrm{u}^{A;BD,C}(\mathrm{E})$ +	$\mathrm{u}^{A;B,CD}(\mathrm{E})$ =	$\mathrm{u}^{A;B,C,D}(\mathrm{E})$ +	$\mathrm{d}_D^{AD;B,C}(\mathrm{E})$
AB;CD	0	1	0	1
AC;BD	1	0	0	1
A;B,C,D	1	1	1	1
A;B,CD	0	1	0	1
A;BD,C	1	0	0	1

i	$\varphi_i(\mathrm{u}^{A;BD,C})$ +	$\varphi_i(\mathrm{u}^{A;B,CD})$ =	$\varphi_i(\mathrm{u}^{A;B,C,D})$ +	$\varphi_i(\mathrm{d}_D^{AD;B,C})$
D	b	b	c	0

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i	$\varphi_i(\mathrm{u}^{A;BD,C})$ +	$\varphi_i(\mathrm{u}^{A;B,CD})$ =	$\varphi_i(\mathrm{u}^{A;B,C,D})$ +	$\varphi_i(\mathrm{d}_D^{AD;B,C})$
D	b	b	c	0

- The partition game value characterization holds on any convex cone of games containing the dummy games

$$d_i^{T;Q}(S;P) = 1 \text{ if } (T \setminus \{i\}; Q[i,R]) \precsim (S;P) \text{ for some } R \in Q \cup \{\emptyset\}.$$

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AB;CD	0	1	0	1
AC;BD	1	0	0	1
A;B,C,D	1	1	1	1
A;B,CD	0	1	0	1
A;BD,C	1	0	0	1

i	$\varphi_i(\mathrm{u}^{A;BD,C})$ +	$\varphi_i(\mathrm{u}^{A;B,CD})$ =	$\varphi_i(\mathrm{u}^{A;B,C,D})$ +	$\varphi_i(\mathrm{d}_D^{AD;B,C})$
D	b	b	c	0

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- Myerson (1977) proved there is a unique additive, symmetric, and carrier value. Carrier is redundant on partition competitive games.

Dispersion Process

Dispersion Process

For coalition games:
ABC

Dispersion Process

For coalition games:

$ABC \underset{1/3}{\Longrightarrow} BC$

Dispersion Process

For coalition games:

$$ABC \underset{1/3}{\Longrightarrow} BC \underset{1/2}{\Longrightarrow} C$$

Dispersion Process

For coalition games:

$$ABC \underset{1/3}{\Longrightarrow} BC \underset{1/2}{\Longrightarrow} C \underset{1}{\Longrightarrow} \emptyset$$

Dispersion Process

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$w(ABC) - w(BC)$ to A	$w(BC) - w(C)$ to B	$w(C)$ to C

Dispersion Process

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$w(ABC) - w(BC)$ to A	$w(BC) - w(C)$ to B	$w(C)$ to C

For partition games:

$$ABC;\emptyset \underset{1/3}{\Longrightarrow} BC;A \underset{\frac{1}{2}a(1;1)}{\Longrightarrow} C;AB \underset{1}{\Longrightarrow} \emptyset$$

$w(ABC;\emptyset) - w(BC;A)$ to A	$w(BC;A) - w(C;AB)$ to B	$w(C;AB)$ to C

$$ABC;\emptyset \underset{1/3}{\Longrightarrow} BC;A \underset{\frac{1}{2}a(1;0)}{\Longrightarrow} C;A,B \underset{1}{\Longrightarrow} \emptyset$$

$w(ABC;\emptyset) - w(BC;A)$ to A	$w(BC;A) - w(C;A,B)$ to B	$w(C;A,B)$ to C

where $a(1;1) + a(1;0) = 1$.

Dispersion and Apex Values

$$
\begin{aligned}
\phi_A^a(w) &= \tfrac{1}{3} && (w(ABC;\emptyset) - w(BC;A)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;1) && (w(AB;C) - w(B;AC)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;0) && (w(AB;C) - w(B;A,C)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;1) && (w(AC;B) - w(C;AB)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;0) && (w(AC;B) - w(C;A,B)) \\
&+ 2\tfrac{1}{3}\tfrac{1}{2}a(1;1) && w(A;BC) \\
&+ 2\tfrac{1}{3}\tfrac{1}{2}a(1;0) && w(A;B,C)
\end{aligned}
$$

Dispersion and Apex Values

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&+ \tfrac{1}{3}\tfrac{1}{2}a(1;0) && (w(AB;C) - w(B;A,C)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;1) && (w(AC;B) - w(C;AB)) \\
&+ \tfrac{1}{3}\tfrac{1}{2}a(1;0) && (w(AC;B) - w(C;A,B)) \\
&+ 2\tfrac{1}{3}\tfrac{1}{2}a(1;1) && w(A;BC) \\
&+ 2\tfrac{1}{3}\tfrac{1}{2}a(1;0) && w(A;B,C)
\end{aligned}
$$

$$
\begin{aligned}
&\phi_A^{12}(w) \\
&= \tfrac{1}{3}(w(ABC;\emptyset) - w(BC;A)) \\
&+ \tfrac{1}{6}(w(AB;C) - w(B;AC)) \\
&+ \tfrac{1}{6}(w(AC;B) - w(C;AB)) \\
&+ \tfrac{1}{3}w(A;BC)
\end{aligned}
$$

$$
\begin{aligned}
&\phi_A^{1,2}(w) \\
&= \tfrac{1}{3}(w(ABC;\emptyset) - w(BC;A)) \\
&+ \tfrac{1}{6}(w(AB;C) - w(B;A,C)) \\
&+ \tfrac{1}{6}(w(AC;B) - w(C;A,B)) \\
&+ \tfrac{1}{3}w(A;B,C)
\end{aligned}
$$

Main Characterization

Theorem (Housman, 1996)

Suppose ϕ is a value on a convex cone Γ containing dummy games. The following statements are equivalent:

1. *ϕ is a Shapley value.*
2. *ϕ is a dispersion value.*
3. *ϕ is an affine combination of apex values.*
4. *ϕ is efficient, symmetric, dummy, and linear on Γ.*

Second Characterization

Monotone $\phi_i(v) \leq \phi_i(w)$ whenever $i \in N$ and $v, w \in \Gamma_n$ satisfy $v(S;P) \leq w(S;P)$ if $i \in S$ and $v(S;P) \geq w(S;P)$ if $i \notin S$.

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Theorem (Housman, 1996)

Suppose ϕ is a value on a convex cone Γ containing dummy games and having a nonempty interior. The following statements are equivalent:

1. *ϕ is a Shapley value with nonnegative coefficients.*
2. *ϕ is a dispersion value with nonnegative parameters.*
3. *ϕ is a convex combination of apex values.*
4. *ϕ is efficient, symmetric, dummy, linear, and monotone on Γ.*

Different Numbers of Players

$S;P$	$w(S;P)$
$ABC;\emptyset$	1
$AB;C$	1
$AC;B$	1
$A;B,C$	1

i	$\phi_i(w)$
A	$1-2\alpha$
B	α
C	α

$\alpha = c_3(1;1)$

Different Numbers of Players

$S;P$	$w(S;P)$
$ABC;\emptyset$	1
$AB;C$	1
$AC;B$	1
$A;B,C$	1

$S;P$	$w^D(S;P)$
$ABCD;\emptyset$	1
$ABC;D$	1
$ABD;C$	1
$AB;C,D$	1
$AB;CD$	1
$ACD;B$	1
$AC;B,D$	1
$AC;BD$	1
$AD;B,C$	1
$A;B,C,D$	1
$A;B,CD$	1
$A;BD,C$	1

i	$\phi_i(w)$
A	$1-2\alpha$
B	α
C	α

$\alpha = c_3(1;1)$

i	$\phi_i(w^D)$
A	$1-2\beta$
B	β
C	β
D	0

Different Numbers of Players

$S;P$	$w(S;P)$
$ABC;\emptyset$	1
$AB;C$	1
$AC;B$	1
$A;B,C$	1

$S;P$	$w^D(S;P)$
$ABCD;\emptyset$	1
$ABC;D$	1
$ABD;C$	1
$AB;C,D$	1
$AB;CD$	1
$ACD;B$	1
$AC;B,D$	1
$AC;BD$	1
$AD;B,C$	1
$A;B,C,D$	1
$A;B,CD$	1
$A;BD,C$	1

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters c_n and c_{n+1}.

i	$\phi_i(w)$
A	$1-2\alpha$
B	α
C	α

$\alpha = c_3(1;1)$

i	$\phi_i(w^D)$
A	$1-2\beta$
B	β
C	β
D	0

Different Numbers of Players

$S;P$	$w(S;P)$	$S;P$	$w^D(S;P)$
$ABC;\emptyset$	1	$ABCD;\emptyset$	1
		$ABC;D$	1
$AB;C$	1	$ABD;C$	1
		$AB;C,D$	1
		$AB;CD$	1
$AC;B$	1	$ACD;B$	1
		$AC;B,D$	1
		$AC;BD$	1
$A;B,C$	1	$AD;B,C$	1
		$A;B,C,D$	1
		$A;B,CD$	1
		$A;BD,C$	1

i	$\phi_i(w)$
A	$1-2\alpha$
B	α
C	α

$\alpha = c_3(1;1)$

i	$\phi_i(w^D)$
A	$1-2\beta$
B	β
C	β
D	0

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters c_n and c_{n+1}.

The *dummy extension* of the partition game (N,w) is the partition game $(N\cup\{d\},w^d)$ defined by $w^d(S;P) = w(S\setminus\{d\};\{R\setminus\{d\}:R\in P\}\setminus\{\emptyset\})$ for all embedded coalitions $(S;P)$ of $N\cup\{d\}$.

Different Numbers of Players

$S;P$	$w(S;P)$
$ABC;\emptyset$	1
$AB;C$	1
$AC;B$	1
$A;B,C$	1

$S;P$	$w^D(S;P)$
$ABCD;\emptyset$	1
$ABC;D$	1
$ABD;C$	1
$AB;C,D$	1
$AB;CD$	1
$ACD;B$	1
$AC;B,D$	1
$AC;BD$	1
$AD;B,C$	1
$A;B,C,D$	1
$A;B,CD$	1
$A;BD,C$	1

i	$\phi_i(w)$
A	$1-2\alpha$
B	α
C	α

$\alpha = c_3(1;1)$

i	$\phi_i(w^D)$
A	$1-2\beta$
B	β
C	β
D	0

The efficient, symmetric, dummy, and linear properties provide no relationship between the parameters c_n and c_{n+1}.

The *dummy extension* of the partition game (N,w) is the partition game $(N \cup \{d\}, w^d)$ defined by $w^d(S;P) = w(S \setminus \{d\}; \{R \setminus \{d\} : R \in P\} \setminus \{\emptyset\}$ for all embedded coalitions $(S;P)$ of $N \cup \{d\}$.

Dummy Independent $\forall w, w^d \in \Gamma$ and $\forall i \in N$, $\phi_i\left(w^d\right) = \phi_i(w)$.

Summary Values

Given a partition game w, define a *summary coalition game* w^b by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|) w(S; P)$$

where

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Theorem

A summary value ϕ^b is a Shapley value on n-player games if and only if

$$b_n(p) = \sum_{r \in p} b_n(p \backslash \{r\} \cup \{r+1\})$$

for all $p \in \mathcal{P}^*(n-2)$.

Third Characterization

Given a partition game w, define a *summary coalition game* w^b by

$$w^b(S) = \sum_{P \in \mathcal{P}(N \setminus S)} b(\|P\|) w(S; P)$$

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$$b(\emptyset) = 1.$$

Define the *summary value* ϕ^b by

$$\phi^b(w) = \mathrm{SH}(w^b).$$

Theorem

Suppose ϕ is a value on a convex cone Γ containing dummy games, having a nonempty interior, and closed with respect to dummy extension. A value ϕ is efficient, symmetric, linear, monotone, and dummy independent on Γ if and only if ϕ is a summary value with nonnegative parameters that are identical for all n and satisfy

$$b(p) = \sum_{r \in p} b(p \setminus \{r\} \cup \{r+1\})$$

for all natural number partitions p.

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