Values for Partially Defined Cooperative Games

David Housman
Mathematics and Computer Science
Goshen College

Undergraduate Student Collaborators:
David Letscher, Notre Dame University (1990)
Thomas Ventrudo, Loyola College of Maryland (1991)
Jodi Wallman, Bryn Mawr College (1991)
Roger Lee, Harvard University (1992)
Jennifer Rich, Drew University (1993)
LeeAnne Brutt, Allegheny College (1994)
Anna Engelsone, Goshen College (1999)
Rachelle Ramer, Goshen College (2003)

Cooperative Game Review

Definition. A *cooperative game* is a set of players $N=\{1,2,\ldots,n\}$ and a worth function w from coalitions (nonempty subsets of players) to real numbers. A cooperative game (N,w) is *zero-monotonic* if $w(S)=w(S-\{i\})+w(\{i\})$ for all $i\in S\subseteq N$.

Example. $N=\{1,2,3\}$ and w is as defined in the table.

S	$\{1,2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$	$\{2\}$	$\{3\}$
$w(S)$	20	14	8	6	2	0	0

Definition. An *allocation method* is a function ϕ from cooperative games to allocations (n-vectors of reals).

Definition. The *Shapley value* is an allocation method defined by

$$\varphi_i(N,w)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\left[w(S)-w(S-\{i\})\right]$$

where s is the size of the coalition S. We use the convention $w(\varnothing)=0$.

Example.

$$\varphi_1(N,w)=\frac{2!0!}{3!}\left[20-6\right]+\frac{1!1!}{3!}\left[14-0\right]+\frac{1!1!}{3!}\left[8-0\right]+\frac{0!2!}{3!}\left[2-0\right]=9$$

Definitions. The allocation method ϕ is

- *efficient* if $\sum_{i \in N} \phi_i(N,w) = w(N)$ for all games (N,w). "All of the potential savings are allocated."
- *symmetric* if $\phi_{\pi(i)}(N, \pi \circ w) = \phi_i(N,w)$ for all games (N,w), permutations π of N, and players $i \in N$, where the worth function $\pi \circ w$ is defined by $(\pi \circ w)(\pi(S)) = w(S)$ for all coalitions S. "A player's name is irrelevant."
- *dummy subsidy-free* if $\phi_i(N,w) = 0$ for all games (N,w) and players $i \in N$ satisfying $w(S) = w(S - \{i\})$ for all coalitions $S \subseteq N$. "Players who never contribute to the worth of any coalition receive nothing."
- *additive* if $\phi_i(N, v+w) = \phi_i(N,v) + \phi_i(N,w)$ for all games (N,v) and (N,w). "Accounting procedures are irrelevant."

Theorem (Shapley, 1953). The Shapley value is the unique allocation method that is efficient, symmetric, dummy subsidy-free, and additive.

S	N	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$	ϕ_1	ϕ_2	ϕ_3
$v(S)$	12	6	6	6	0	4	4	4
$u^{12}(S)$	6	6	0	0	0	3	3	0
$u^{1}(S)$	2	2	2	0	2	2	0	0
$w(S)$	20	14	8	6	2	9	7	4

Partially Defined Cooperative Games

Definition. A *partially defined cooperative game (PDG)* is a set of players $N=\{1,2,\ldots,n\}$, a collection of coalitions $\mathcal{C}$ containing N, and a worth function w from $\mathcal{C}$ to reals.

Example. $N=\{1,2,3,4,5\}$, $\mathcal{C}=\{S\subseteq N:|S|\in\{1,4,5\}\}$, and w is as defined in the table.

S	12345	1234	1235	1245	1345	2345	i
$w(S)$	600	480	480	360	180	60	0

Definition. An *extension* of the PDG $(N,\mathcal{C},w)$ is a game $(N,\hat{w})$ satisfying $\hat{w}(S)=w(S)$ for all $S\in\mathcal{C}$.

Example. $N=\{1,2,3,4,5\}$ and $\hat{w}$ is as defined in the table.

S	12345	1234	1235	1245	1345	2345	i
$\hat{w}(S)$	600	480	480	360	180	60	0

S	123	124	125	134	135	145	234	235	245	345
$\hat{w}(S)$	300	240	240	120	120	120	40	40	40	40

S	12	13	14	15	23	24	25	34	35	45
$\hat{w}(S)$	120	60	60	60	20	20	20	20	20	20

Definition. A PDG $(N,\mathcal{C},w)$ is *zero-monotonic* if it has a zero-monotonic extension.

Definitions. The allocation method ϕ is

- *efficient* if $\sum_{i \in N} \phi_i(N, \mathcal{C}, w) = w(N)$ for all PDGs $(N, \mathcal{C}, w)$.
- *symmetric* if $\phi_{\pi(i)}(N, \pi \circ \mathcal{C}, \pi \circ w) = \phi_i(N, \mathcal{C}, w)$ for all PDGs $(N, \mathcal{C}, w)$, permutations π of N, and players $i \in N$.
- *dummy subsidy-free* if $\phi_i(N, \mathcal{C}, w) = 0$ for all PDGs $(N, \mathcal{C}, w)$ and players $i \in N$ satisfying $\underline{\hat{w}(S) = \hat{w}(S - \{i\})}$ for all coalitions $S \subseteq N$ and zero-monotonic extensions $\underline{(N, \hat{w})}$.
- *additive* if $\phi_i(N, \mathcal{C}, v + w) = \phi_i(N, \mathcal{C}, v) + \phi_i(N, \mathcal{C}, w)$ for all PDGs $(N, \mathcal{C}, v)$ and $(N, \mathcal{C}, w)$.

Theorem (Housman, 2001). The Shapley value (with 0s substituted for the unknown coalitional worths) is the unique allocation method on zero-monotonic PDGs that is efficient, symmetric, dummy subsidy-free, and additive.

S	N	1234	1235	1245	1345	2345	ϕ_1	2	3	4	5
$u^1(S)$	60	0	0	0	0	60	0	15	15	15	15
$u^2(S)$	180	0	0	0	180	0	45	0	45	45	45
$u^3(S)$	360	0	0	360	0	0	90	90	0	90	90
$u^4(S)$	480	0	480	0	0	0	120	120	120	0	120
$u^5(S)$	480	480	0	0	0	0	120	120	120	120	0

$u^0(S)$	960	0	0	0	0	0	192	192	192	192	192
$w(S)$	600	480	480	360	180	60	183	153	108	78	78

Why not use the Shapley Value?

Five-player PDG:

S	12345	1234	1235	1245	1345	2345	i
$w(S)$	5	4	3	2	1	0	0

Any zero-monotonic extension satisfies

S	123	124	125	134	135	145	234	235	245	345
$\hat{w}(S)$	≤3	≤2	≤2	≤1	≤1	≤1	0	0	0	0

S	12	13	14	15	23	24	25	34	35	45
$\hat{w}(S)$	0≤	0≤	0≤	0≤	0	0	0	0	0	0

$$0 \le \hat{w}(ij) \le \hat{w}(ijk) \le w(ijkl)$$

The Shapley Value of w equals the Shapley Value of the extension $\hat{w}$ for which $\hat{w}(ij) = \hat{w}(ijk) = 0$.

This is the apex of the "pyramid" of extensions.

It gives player 1 the minimum payoff among all possible extension Shapley values.

Additivity is too strong of a property because we ask for the allocations to add even when the sets of extensions do not.

Weakly Additive Values

Definitions. The allocation method ϕ is

- *weakly additive* if $\phi_i(N,\mathcal{C},v+w)=\phi_i(N,\mathcal{C},v)+\phi_i(N,\mathcal{C},w)$ whenever $\text{ext}(N,\mathcal{C},v)+\text{ext}(N,\mathcal{C},w)=\text{ext}(N,\mathcal{C},v+w)$.
- *proportional* if $\phi_i(N,\mathcal{C},aw)=a\phi_i(N,\mathcal{C},w)$ for all real numbers a and PDGs $(N,\mathcal{C},w)$.

Theorem. Suppose $\mathcal{C}=\{S\subseteq N:|S|\in\{1,n-1,n\}\}$ and $\text{ext}(w)$ is the set of zero-monotonic extensions of the PDG $(N,\mathcal{C},w)$. Then $\text{ext}(v)+\text{ext}(w)=\text{ext}(v+w)$ if and only if there exists a permutation π of N such that
$v(N-\{\pi(1)\})\le v(N-\{\pi(2)\})\le\cdots\le v(N-\{\pi(n)\})$ and
$w(N-\{\pi(1)\})\le w(N-\{\pi(2)\})\le\cdots\le w(N-\{\pi(n)\})$.

Suppose $w(1234)\ge w(1235)\ge w(1245)\ge w(1345)\ge w(2345)$.

12345	1234	1235	1245	1345	2345	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	1	1	1	1	1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
1	1	1	1	1	0	$1-a_4$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$	$\frac{1}{4}a_4$
1	1	1	1	0	0	$\frac{1}{2}(1-a_3)$	$\frac{1}{2}(1-a_3)$	$\frac{1}{3}a_3$	$\frac{1}{3}a_3$	$\frac{1}{3}a_3$
1	1	1	0	0	0	$\frac{1}{3}(1-a_2)$	$\frac{1}{3}(1-a_2)$	$\frac{1}{3}(1-a_2)$	$\frac{1}{2}a_2$	$\frac{1}{2}a_2$
1	1	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
1	0	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

In the following, assume that $\mathcal{C} = \{S \subseteq N : |S| \in \{1, n-1, n\}\}$ and all extensions are zero-monotonic.

Definition. Suppose $M \subseteq N$. Define the PDG v^M by $v^M(N) = v^M(N - \{i\}) = 1$ if $i \in M$ and $v^M(S) = 0$ otherwise.

Theorem. If ϕ is efficient and symmetric, then there are $a_0, a_1, \ldots, a_n$ satisfying

$$\phi_i(v^M) = \begin{cases} \frac{1}{m} a_m, & \text{if } i \in M \\ \frac{1}{n-m}(1 - a_m), & \text{if } i \notin M \end{cases}$$

where $a_0 = 0$ and $a_n = 1$. If ϕ is also dummy subsidy-free, then $a_1 = 0$. If ϕ is also additive, then $a_m = \frac{m(m-1)}{n(n-1)}$ for all m.

Theorem. The allocation method ϕ is efficient, symmetric, dummy subsidy-free, proportional, and weakly additive if and only if

$$\begin{aligned} \phi_{\pi(i)}(w) = {} & \tfrac{1}{n} w(N - \{\pi(1)\}) \\ & + \sum_{j=2}^{i} \frac{1}{n-j+1} a_{n-j+1} [w(N - \{\pi(j)\}) - w(N - \{\pi(j-1)\})] \\ & + \sum_{j=i+1}^{n} \frac{1}{j-1} (1 - a_{n-j+1}) [w(N - \{\pi(j)\}) - w(N - \{\pi(j-1)\})] \\ & + \tfrac{1}{n} [w(N) - w(N - \{\pi(n)\})] \end{aligned}$$

Geometric Approach

Definition. Let e^T be the game satisfying $e^T(T)=1$ and $e^T(S)=0$ for all $S \neq T$.

Definition. Suppose w is a PDG. The extension $\hat{w}$ is a *coordinate center* of ext(w) if $\hat{w}$ is the midpoint of the line segment $\{\hat{w}+\lambda e^T : \lambda \in \mathbb{R}\} \cap \text{ext}(w)$ for all coalitions $T \subseteq N$.

Theorem (Brutt, 1994). For each zero-monotonic PDG, a coordinate center exists and is unique.

Example. The PDG

S	12345	1234	1235	1245	1345	2345	i
$w(S)$	600	480	480	360	180	60	0

has coordinate center

S	123	124	125	134	135	145	234	235	245	345
$\hat{w}(S)$	300	240	240	120	120	120	40	40	40	40

S	12	13	14	15	23	24	25	34	35	45
$\hat{w}(S)$	120	60	60	60	20	20	20	20	20	20

Coordinate Center Value

Definition. The *coordinate center value* κ is defined by $\kappa(w)$ is the Shapley value of the coordinate center of $\text{ext}(w)$.

Theorem (Brutt, 1994). The coordinate center value is given by the formula

$$\begin{aligned}\phi_{\pi(i)}(w) &= \tfrac{1}{n} w(N-\{\pi(1)\}) \\ &+ \sum_{j=2}^{i} \tfrac{1}{n-j+1} a_{n-j+1}[w(N-\{\pi(j)\})-w(N-\{\pi(j-1)\})] \\ &+ \sum_{j=i+1}^{n} \tfrac{1}{j-1}(1-a_{n-j+1})[w(N-\{\pi(j)\})-w(N-\{\pi(j-1)\})] \\ &+ \tfrac{1}{n}[w(N)-w(N-\{\pi(n)\})]\end{aligned}$$

where $a_m = \dfrac{m-1}{n(n-m+1)}$ for $m=1,2,\ldots,n-2$ and $a_{n-1} = \dfrac{n-1}{2n}$.

Thus, it is an efficient, symmetric, dummy subsidy-free, proportional, and weakly additive allocation method.

Example. The PDG

S	12345	1234	1235	1245	1345	2345	i
$w(S)$	600	480	480	360	180	60	0

has the coordinate center value (224, 164, 94, 59, 59).

Compare to the Shapley value (183, 153, 108, 78, 78).

Other Centers

Centroid:

- The center of mass of $\text{ext}(w)$.
- Not weakly additive and difficult to compute.

Coordinate Extrema Center:

- $\hat{w}(S)=\frac{1}{2}\left[\min\{\hat{u}(S):\hat{u}\in ext(w)\}+\max\{\hat{u}(S):\hat{u}\in ext(w)\}\right]$
- Theorem (Brutt, 1994). The coordinate extrema center $\hat{w}$ of $\text{ext}(w)$ satisfies $\hat{w}(S)=\frac{1}{2}\min\{w(T):S\subseteq T\}$.
- So, the Shapley value of the coordinate extrema center is an efficient, symmetric, subsidy free, proportional, and weakly additive allocation method.

Chebyshev Center:

- The center of the smallest hypersphere containing $\text{ext}(w)$.
- Theorem (Engelsone, 1999). The Chebyshev center of $\text{ext}(w)$ is the coordinate extrema center of $\text{ext}(w)$.

Comparison

N	1234	1235	1245	1345	2345	Payoff to Strong Players				
						Shapley	Centroid	Chebyshev	Coordinate	Max for 1
1	1	1	1	1	0	.400	.583	.600	.600	.800
1	1	1	1	0	0	.350	.420	.425	.433	.500
1	1	1	0	0	0	.300	.317	.317	.317	.333
1	1	0	0	0	0	.250	.250	.250	.250	.250
1	0	0	0	0	0	.200	.200	.200	.200	.200

N	1234	1235	1245	1345	2345	Payoff to Weak Players				
						Shapley	Centroid	Chebyshev	Coordinate	Max for 1
1	1	1	1	1	1	.200	.200	.200	.200	.200
1	1	1	1	1	0	.150	.104	.100	.100	.050
1	1	1	1	0	0	.100	.053	.050	.045	.000
1	1	1	0	0	0	.050	.025	.025	.025	.000
1	1	0	0	0	0	.000	.000	.000	.000	.000

Comparison

N	1234	1235	1245	1345	2345	Shapley Value				
						ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	1	1	1	1	1	.20	.20	.20	.20	.20
1	1	1	1	1	0	.40	.15	.15	.15	.15
1	1	1	1	0	0	.35	.35	.10	.10	.10
1	1	1	0	0	0	.30	.30	.30	.05	.05
1	1	0	0	0	0	.25	.25	.25	.25	.00

N	1234	1235	1245	1345	2345	Coordinate Center Value				
						ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	1	1	1	1	1	.200	.200	.200	.200	.200
1	1	1	1	1	0	.600	.100	.100	.100	.100
1	1	1	1	0	0	.433	.433	.044	.044	.044
1	1	1	0	0	0	.317	.317	.317	.025	.025
1	1	0	0	0	0	.250	.250	.250	.250	.000

N	1234	1235	1245	1345	2345	Shapley		Coord Center	
						strong	weak	strong	weak
1	1	1	1	1	1		.20		.200
1	1	1	1	1	0	.40	.15	.600	.100
1	1	1	1	0	0	.35	.10	.433	.044
1	1	1	0	0	0	.30	.05	.317	.025
1	1	0	0	0	0	.25	.00	.250	.000

Other Past Results and Future Work

Generalize zero-monotonic PDGs on $\{S \subseteq N : |S| \in \{1, n-1, n\}\}$ to arbitrary $\mathcal{C}$.

Axiomatically characterize the coordinate center value.

Consider other types of extensions, including superadditive and convex games.

Consider other types of centers, including centroid, extreme coordinate center, and Chebyshev center.