Fair Division with Money

by

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Inheritance Problem

Bob, Carol, and Doug have inherited equal shares in their mother’s estate consisting of an old cabin on an acre of land in the country, sterling silverware for twelve, and a two year old sports car. The siblings have different opinions about the worth of each item as given in the table below (numbers are in thousands of dollars) and they are willing to either give or receive cash to help obtain a fair allocation of the estate. Suggest a solution.

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Efficient Allocations

An allocation is *efficient* if there is no other allocation that is better for everyone.

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

If each sibling is given one-third of each item, then

Bob values his bundle at $v_B = 54/3 = \$18$.

Carol values her bundle at $v_C = 60/3 = \$20$.

Doug values his bundle at $v_D = 21/3 = \$7$.

If we give Bob the cabin, Carol the car, and Doug the silver, then the values are $\bar{v}_B = \$30$, $\bar{v}_C = \$25$, and $\bar{v}_D = \$8$.
So, the first allocation is <u>not</u> efficient.

Theorem. An allocation is efficient if and only if each item is given to the sibling who values it the most.

First Price Auction

Sell each item to the highest bidder, who pays the amount bid, and the money collected is divided equally among the siblings.

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Bob receives the cabin and pays $30.

Carol receives the silver and car and pays 20 + 25 = $45.

Doug receives and pays nothing.

The 30 + 45 + 0 = $75 in the pot is divided evenly.

Bob receives the cabin and pays 30 – 25 = $5. Bob values this at 30 – 5 = $25.

Carol receives the silver and car and pays 45 – 25 = $20. Carol values this at 20 + 25 – 20 = $25.

Doug receives $25. Doug values this at $25.

Second Price Auction

Sell each item to the highest bidder, who pays the <u>second highest</u> amount bid, and the money collected is divided equally among the siblings.

	Bob	Carol	Doug
Cabin	(30)	15	5
Silver	5	(20)	8
Car	19	(25)	8
Total	54	60	21

Bob receives the cabin and pays $15.

Carol receives the silver and car and pays 8 + 19 = $27.

Doug receives and pays nothing.

The 15 + 27 + 0 = $42 in the pot is divided evenly.

Bob receives the cabin and pays 15 – 14 = $1. Bob values this at 30 – 1 = $29.

Carol receives the silver and car and pays 27 – 14 = $13. Carol values this at 20 + 25 – 13 = $32.

Doug receives $14. Doug values this at $14.

Knaster's Method

Sell each item to the highest bidder and distribute money so that each sibling receives the same incremental value above their "fair share" of the estate (1/3 of what they think the estate is worth).

	Bob	Carol	Doug
Cabin	(30)	15	5
Silver	5	(20)	8
Car	19	(25)	8
Total	54	60	21

The value obtained by each sibling should be

$$v_B = \frac{54}{3} + \lambda = 18 + \lambda \quad v_C = \frac{60}{3} + \lambda = 20 + \lambda \quad v_D = \frac{21}{3} + \lambda = 7 + \lambda$$

Adding these equalities, we obtain

$$75 = v_B + v_C + v_D = 45 + 3\lambda \quad \Rightarrow \quad \lambda = 10$$

Bob's bundle should be worth 18 + 10 = $28: receipt of the cabin and a payment of $2.

Carol's bundle should be worth 20 + 10 = $30: receipt of the silver and car and a payment of $15.

Doug's bundle should be worth 7 + 10 = $17: receipt of $17.

Equitable and Efficient Method

Sell each item to the highest bidder and distribute money so that each sibling receives the same fraction of the estate (from their own perspectives).

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Each sibling should receive the same fraction of the estate:
$v_B / 54 = v_C / 60 = v_D / 21 = \lambda$
Adding these equalities, we obtain
$75 = v_B + v_C + v_D = 54\lambda + 60\lambda + 21\lambda = 135\lambda \quad \Rightarrow \quad \lambda = 5/9$

Bob's bundle should be worth $(5/9)54 = \$30$: receipt of the cabin.

Carol's bundle should be worth $(5/9)60 = \$33.33$: receipt of the silver and car and a payment of \$11.67.

Doug's bundle should be worth $(5/9)21 = \$11.67$: receipt of \$11.67.

Moulin-Shapley Method

Sell each item to the highest bidder and distribute money so that each sibling receives his or her average, over sibling orders, marginal contribution to the group stand-alone values.

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Group	BCD	BC	BD	CD	B	C	D
Stand-Alone Value	75	75	57	60	54	60	21

Order	Bob	Carol	Doug
BCD	54	21	0
BDC	54	18	3
CBD	15	60	0
CDB	15	60	0
DBC	36	18	21
DCB	15	39	21
Total	189	216	45
Average	31.5	36	7.5

Group Stand-Alone Allocations

No group of siblings receives more than its stand-alone value (maximum obtainable if the group owned the entire estate).

	Bob	Carol	Doug
Cabin	(30)	15	5
Silver	5	(20)	8
Car	19	(25)	8
Total	54	60	21

Group	Stand-Alone Value	Constraint
BCD	75	$v_B + v_C + v_D = 75$ if efficient
BC	75	$v_B + v_C \le 75 \quad \Leftrightarrow \quad v_D \ge 0$
BD	57	$v_B + v_D \le 57 \quad \Leftrightarrow \quad v_C \ge 18$
CD	60	$v_C + v_D \le 60 \quad \Leftrightarrow \quad v_B \ge 15$
B	54	$v_B \le 54$
C	60	$v_C \le 60$
D	21	$v_D \le 21$

The first-price action does <u>not</u> yield a group stand-alone allocation for our example because $v_D = 25$.

Group Rational Allocations

Each group of siblings receives at least its ownership value (maximum obtainable if the group owned a proportionate share of the estate).

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Group	Stand Alone	Owner-ship	Constraint
BCD	75	75	$v_B + v_C + v_D = 75$ if efficient
BC	75	50	$v_B + v_C \geq 50 \iff v_D \leq 25$
BD	57	38	$v_B + v_D \geq 38 \iff v_C \leq 37$
CD	60	40	$v_C + v_D \geq 40 \iff v_B \leq 35$
B	54	18	$v_B \geq 18$
C	60	20	$v_C \geq 20$
D	21	7	$v_D \geq 7$

Envy-Free Allocations

No sibling would prefer another's bundle to his or her own.

	Bob	Carol	Doug
Cabin	30	15	5
Silver	5	20	8
Car	19	25	8
Total	54	60	21

Let m_i be the money allocated to sibling i.

Bob does not envy Carol.	$v_B = 30 + m_B \geq 24 + m_C$
	$v_B \geq v_C - 21$
Carol does not envy Bob.	$v_C = 45 + m_C \geq 15 + m_B$
	$v_C \geq v_B - 15$
Bob does not envy Doug.	$v_B = 30 + m_B \geq m_D$
	$v_B \geq v_D$
Doug does not envy Bob.	$v_D = m_D \geq 5 + m_B$
	$v_D \geq v_B - 25$
Carol does not envy Doug.	$v_C = 45 + m_C \geq m_D$
	$v_C \geq v_D$
Doug does not envy Carol.	$v_D = m_D \geq 16 + m_C$
	$v_D \geq v_C - 29$

Incompatibilities of the Properties

	Bob	Carol	Doug
Item	30	27	3

Consider an efficient allocation.

If the allocation is group stand-alone, then $v_D \le 3$.

If the allocation is envy-free, then $v_B = 30 + m_B \ge m_C, m_D$, $v_C = m_C \ge 27 + m_B, m_D$, , and $v_D = m_D \ge 3 + m_B, m_C$. So, $m_C = m_D = m$. Since $m_B + m_C + m_D = 0$, we have $30 - 2m \ge m$ and $m \ge 27 - 2m$. Since $m = v_D$, these imply $9 \le v_D \le 10$. Hence, the allocation cannot both be enny-free and group stand-alone.

If the allocation is group rational, then $v_B + v_D \ge 20$ and $v_C + v_D \ge 18$. Adding these inequalities together, we obtain $30 + v_D = v_B + v_C + v_D + v_D \ge 38$, and so $v_D \ge 8$. Hence, the allocation cannot both be group rational and group stand-alone.

Properties of Allocation Methods (Part 1)

	Bob	Carol	Doug
Item	30	27	3

If the allocation is group stand-alone, then $v_D \le 3$.
If the allocation is efficient and envy-free, then $9 \le v_D \le 10$.
If the allocation is efficient and group rational, $v_D \ge 8$.

The first price auction obtains $v = (10,10,10)$, and so is not group stand-alone.

The second price auction obtains $v = (12,9,9)$, and so is not group stand-alone.

Knaster's obtains $v = (10+\lambda, 9+\lambda, 1+\lambda) = (13.3, 12.3, 4.3)$, and so is not group stand-alone, envy-free, or group rational.

Equitable obtains $v = (30\lambda, 27\lambda, 3\lambda) = (15, 13.5, 1.5)$, and so is not envy-free or group rational.

Moulin-Shapley obtains $v = (16,13,1)$, and so is not envy-free or group rational.

Properties of Allocation Methods (Part 2)

	Bob	Carol	Doug
Item	30	10	10

Equitable obtains $v = (30\lambda, 10\lambda, 10\lambda) = (18, 6, 6)$, and so $v_B + v_C = 12$.

The stand-alone value for the group consisting of Carol and Doug is 10, and so the equitable method is not group stand-alone.

Summary

	Envy-Free	Group Rational	Group Stand-Alone
First Price Auction	Yes	Yes	No
Second Price Auction	Yes	Yes	No
Knaster's Method	No	No	No
Equitable and Efficient	No	No	No
Moulin-Shapley	No	No	Yes