# A Beautiful Mind:
## Some Game Theory of John Nash

**David Housman**
**Goshen College**
**Indiana MAA Student Workshop**

# Nash References

Biography

Sylvia Nasar, *A Beautiful Mind*, Simon and Schuster, 1999.

Academic Papers

Harold W. Kuhn and Sylvia Nasar (eds.), *The Essential John Nash*, Princeton University Press, 2002.

Hex

John Milnor, A Nobel Prize of John Nash, *The Mathematical Intelligencer* 17, no. 3 (1995): 11-17.

Strategic Game Equilibrium

John Nash, Equilibrium Points in *n*-Person Games, *Proceedings of the National Academy of Sciences* 36 (1950): 48-49.

Bargaining Solution

John Nash, The Bargaining Problem, *Econometrica* 18 (1950): 155-162.

**Game Theory:** the mathematical study of situations involving conflict and/or cooperation.

# "Nash" or Hex

There are two players: circle and square. Circle moves first by “capturing” (placing a circle around) a single dot anywhere in the rhombus. Square moves second by “capturing” (placing a square around) a single uncaptured dot. The players continue to take turns capturing previously uncaptured dots (*not* necessarily adjacent to previously captured dots). A player wins if the dots s/he has captured include a “connected” set of adjacent dots linking his or her two sides. Two dots are *adjacent* if each is closest to the other; thus each dot in the interior has six adjacent dots.

```
                  circle

            *   *   *   *   *

          *   *   *   *   *

square  *   *   *   *   *  square

      *   *   *   *   *

    *   *   *   *   *

          circle
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# 2-Hex Winning Strategy

# 3-Hex Winning Strategy

# 4-Hex Winning Strategy

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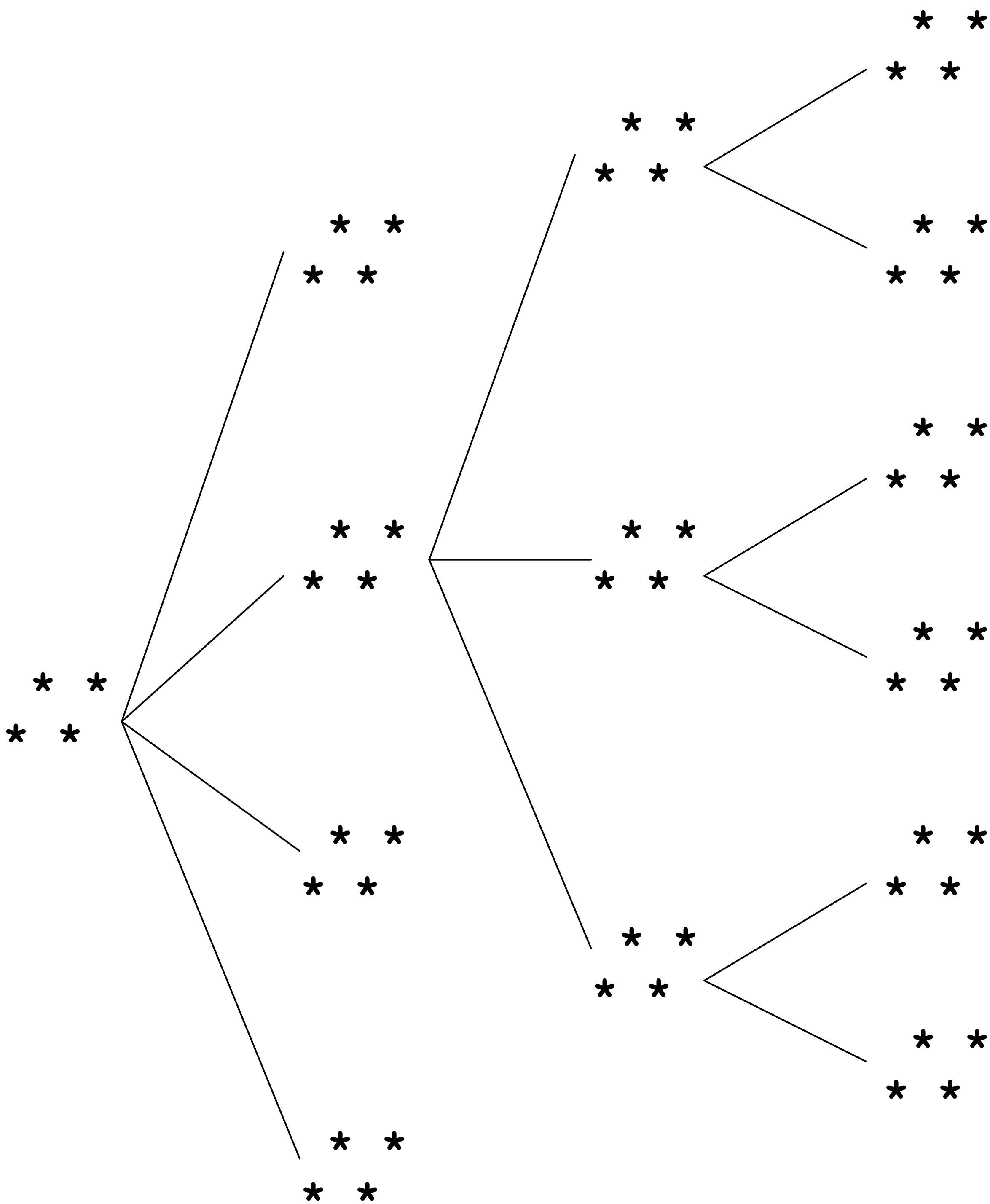
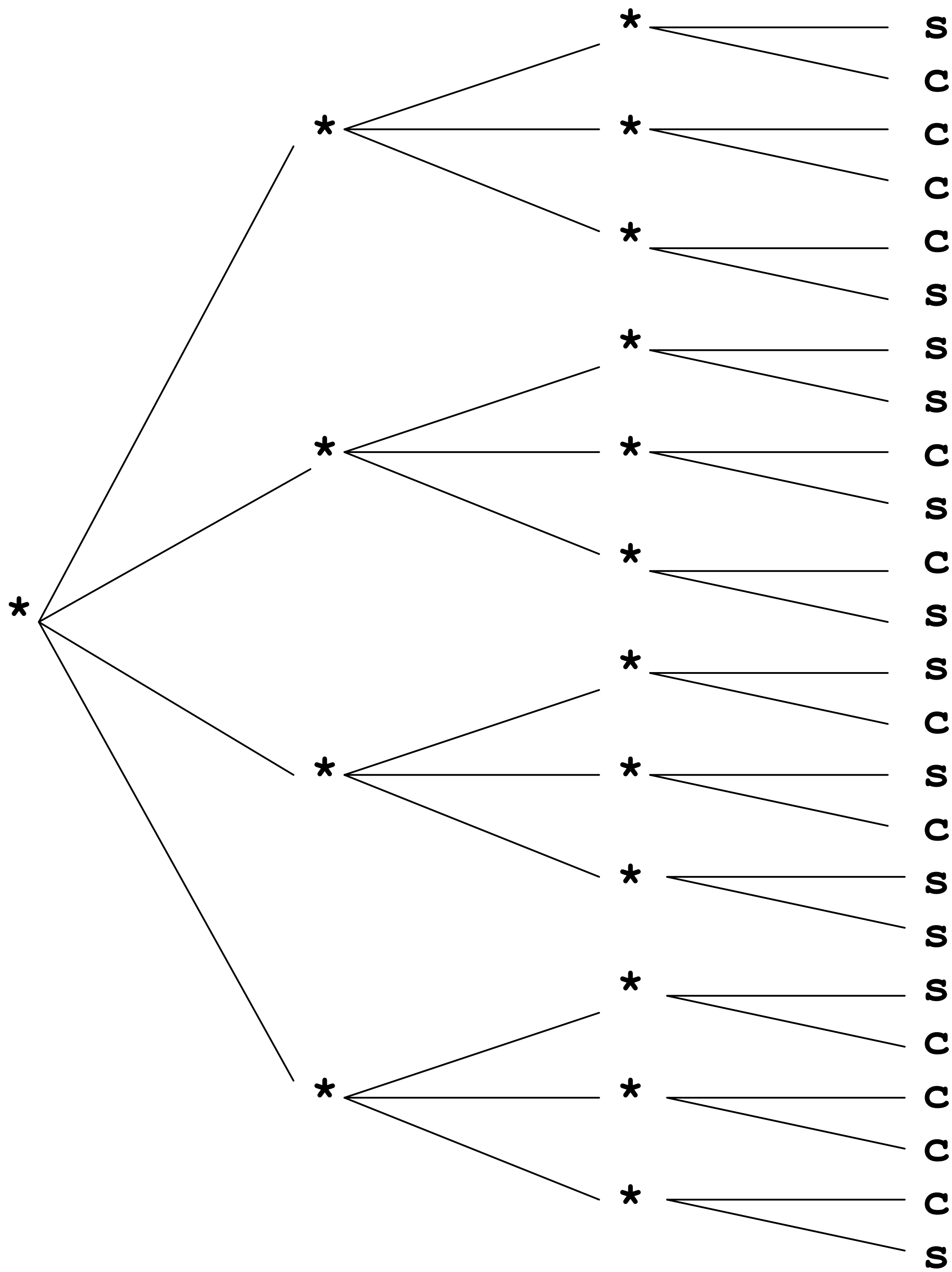
# 2-Hex Game Tree Construction



# 2-Hex Game Tree Backward Induction



# Zermelo's Theorem

Suppose a game

- involves two players,
- has perfect information,
- has a finite number of possible moves, and
- always ends in one player winning.

Then one of the players can always force a win.

Chess

- involves two players,
- has perfect information,
- has a finite number of possible moves, and
- always ends in one player winning <u>or a draw</u>.

Either white can force a win, or black can force a win, <u>or both sides can force at least a draw</u>.

So why is chess so hard to play?

# Nash's Result
# Circle Can Always Force a Win in Hex

Suppose square can always force a win in Hex.

Define *Reversed Hex* to be the same as Hex, except that square moves first.

By our supposition and the symmetry of the game board, circle can always force a win in Reversed Hex.

Consider now a play of Hex.  Let circle capture an arbitrary point, subsequently ignoring that move and playing to win as if in Reversed Hex.  If at any time circle is suppose to capture the point already captured on the first move,  let circle capture another arbitrary point.

Circle must win!

This contradicts our supposition, and so square cannot force a win in Hex.

By Zermelo's Theorem, circle can always force a win in Hex.

Notice that we have only proved existence.  An open question is to describe how to force the win.

**Game Theory Workshop**
**David Housman, Goshen College,** dhousman@goshen.edu, **www.goshen.edu/~dhousman**

**Nash or Hex**

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# Silence!

## Get ready to play another game.

In Hex, one player wins and one player loses. The game is "zero sum." There is no reason for cooperation.

In the next game, just like in real life, there are opportunities for cooperation. Both players can "win" and both can "lose" – by possibly different amounts. This makes the situation more complex.

We will simplify the situation by giving each player only three strategies and not permit agreements, negotiation, arbitration, or even communication.

The handout will explain the game in writing, and I will explain the game orally. I will answer clarification questions. Otherwise, let there be silence during this game.

# A Strategic Game

Secretly, Rose writes A, B, or C. Secretly, Colin writes D, E, or F. They simultaneously reveal what they wrote. They find the cell in the following table corresponding to their choices. The first number is the payoff to Rose, and the second number is the payoff to Colin. For example, if Rose wrote A and Colin wrote F, then Rose receives 8 and Colin receives 2.

| | D | E | F |
|---|---|---|---|
| A | 1, 3 | 6, 4 | 8, 2 |
| B | 5, 7 | 4, 8 | 9, 1 |
| C | 2, 9 | 5, 6 | 7, 5 |

You will play this game twice: once as Rose and once as Colin. After everyone has made their choices, Rose choices will be randomly paired with Colin choices. Please do not communicate with anyone else.

# The Game Payoffs

I told a white lie: the Rose choices and Colin choices will not actually be randomly paired.  Instead, we will determine the expected payoffs for each strategy.  First, let's find out how many chose each strategy by a show of hands.

| | count | | | | |
|---|---|---|---|---|---|
| count | | D | E | F | payoff |
| | A | 1, 3 | 6, 4 | 8, 2 | |
| | B | 5, 7 | 4, 8 | 9, 1 | |
| | C | 2, 9 | 5, 6 | 7, 5 | |
| | payoff | | | | |

Let's calculate the expected payoff for choosing an A:

# Dominance

| | D | E | F |
|---|---|---|---|
| A | 1, 3 | 6, 4 | 8, 2 |
| B | 5, 7 | 4, 8 | 9, 1 |
| C | 2, 9 | 5, 6 | 7, 5 |

Strategy S dominates strategy T if for each choice of strategies by the other players, strategy S provides a payoff at least as great as the payoff provided by strategy T.

# Nash Equilibrium

| | D | E | F |
|---|---|---|---|
| A | 1, 3 | 6, 4 | 8, 2 |
| B | 5, 7 | 4, 8 | 9, 1 |
| C | 2, 9 | 5, 6 | 7, 5 |

A Nash equilibrium is a function from players to strategies for which no player has a unilateral incentive to deviate.

# Battle of the Sexes

Rose and Colin would like to share a campus event with each other this evening but are unable to contact each other.  There will be a concert and a sports event.  Rose is primarily interested in being with Colin but would prefer the concert over the sports event.  Colin is primarily interested in being with Rose but would prefer the sports event over the concert.  Analyze.

# Matching Pennies

In secret, Rose and Colin each choose "heads" or "tails". When they reveal their choices, Rose receives $1 from Colin if the choices match and Colin receives $1 from Rose if the choices do not match.  Analyze.

# Nash Equilibrium

**Theorem.** Every strategic game with a finite number of players, each of whom has a finite number of pure strategies, has at least one Nash equilibrium in pure or mixed strategies.

Graphical Proof for Two Players Each with Two Pure Strategies:

# Nash Equilibrium and Efficiency

| | D | E | F |
|---|---|---|---|
| A | 1, 3 | 6, 4 | 8, 2 |
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**Non-cooperative game:** players act on their own without collaboration or communication with any of the others.

**Cooperative game:** players have opportunities to share information, make deals, and join coalitions.

**Negotiation and Arbitration:** If Rose and Colin could make a binding agreement with each other, what would or should they agree to do?

# Potential Payoffs

| | D | E | F |
|---|---|---|---|
| **A** | 1, 3 | 6, 4 | 8, 2 |
| **B** | 5, 7 | 4, 8 | 9, 1 |
| **C** | 2, 9 | 5, 6 | 7, 5 |

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# Potential Payoffs

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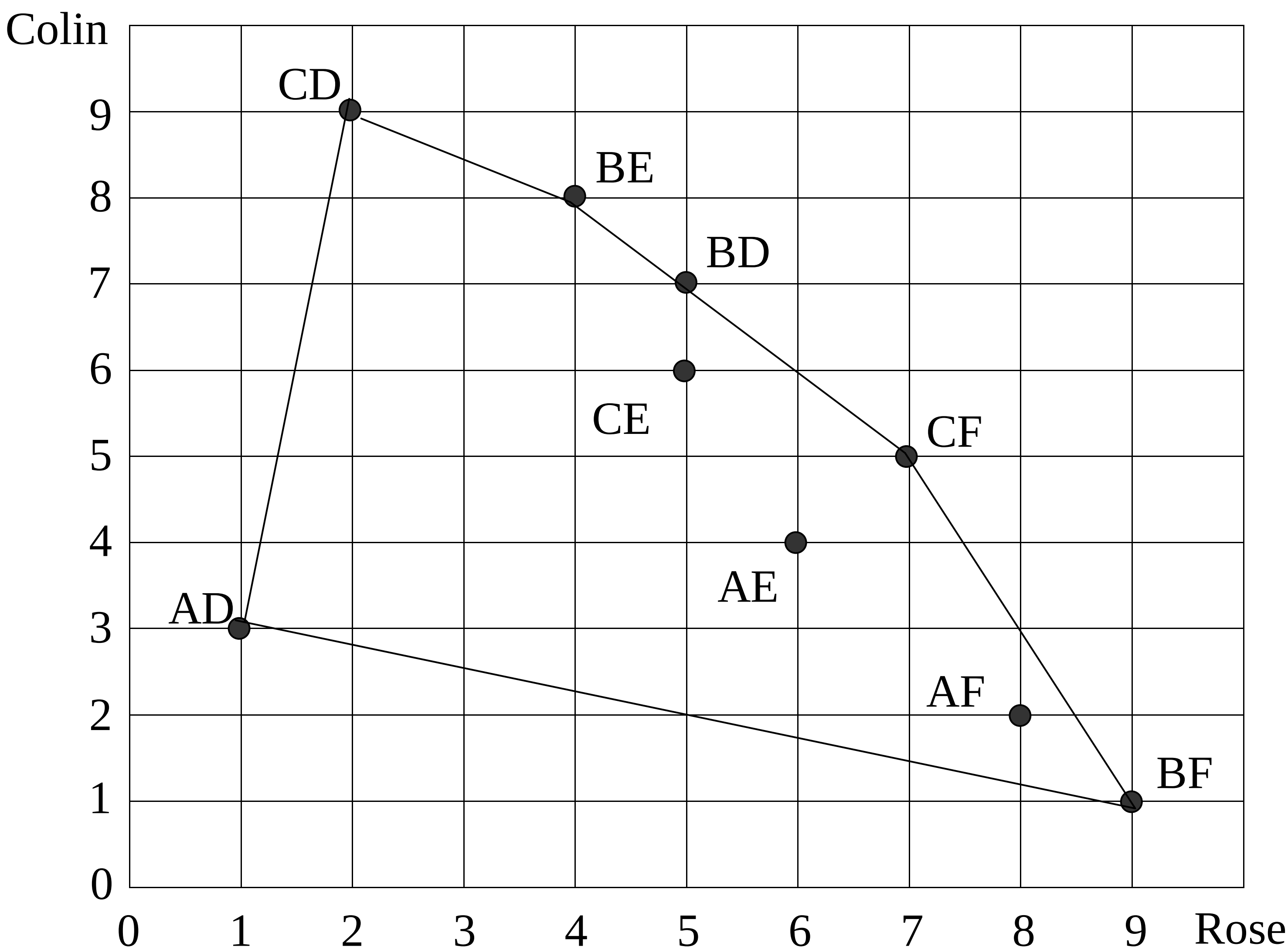

# Nash's Bargaining Solution

Let $u_1$ and $u_2$ be utility functions for the two players. Let $c(S)$ represent the solution point in a set $S$ which is compact and convex and included the origin. We assume:

Efficiency: If $\alpha$ is a point in $S$ such that there exists another point $\beta$ in $S$ with the property $u_1(\beta) > u_1(\alpha)$ and $u_2(\beta) > u_2(\alpha)$, then $\alpha \neq c(S)$.

Equal Treatment: If $S$ is symmetric ($(a, b) \in S \Rightarrow (b, a) \in S$), then $c(S)$ is a point of the form $(c, c)$.

Independence of Irrelevant Alternatives: If the set $T$ contains the set $S$ and $c(T)$ is in $S$, then $c(T) = c(S)$.

**Theorem.** If $c$ is a bargaining solution that satisfies the efficiency, equal treatment, and independence of irrelevant alternatives properties, then $c(S)$ is the point of $S$ in the first quadrant that maximizes $u_1u_2$.

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You will play this game twice: once as Rose and once as Colin. After everyone has made their choices, Rose choices will be randomly paired with Colin choices. Please do not communicate with anyone else.

Write your choices here. Rose: __________ Colin: __________


| | D | E | F |
|---|---|---|---|
| A | 1, 3 | 6, 4 | 8, 2 |
| B | 5, 7 | 4, 8 | 9, 1 |
| C | 2, 9 | 5, 6 | 7, 5 |

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