Mathematics of Congressional Apportionment

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The Constitutional Basis

"Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers . . . . The actual Enumeration shall be made within three years after the first meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such manner as they shall by Law direct."

article I, section 2

What is the Problem?

$$\text{quota}_{\text{CA}} = \frac{\text{population of CA}}{\text{population of USA}} \times \text{house size}$$

$$= \frac{33{,}930{,}798}{281{,}424{,}177} \times 435 = 52.447$$

$$\text{quota}_{\text{UT}} = \frac{\text{population of UT}}{\text{population of USA}} \times \text{house size}$$

$$= \frac{2{,}236{,}714}{281{,}424{,}177} \times 435 = 3.457$$

The above figures are from the 2000 census, and the official apportionment is

$$\text{apportionment}_{\text{CA}} = 53$$

$$\text{apportionment}_{\text{UT}} = 3$$

A Small Example

State i	Population p_i	Quota q_i	Apportionment a_i
1	9,598	47.99	
2	5,868	29.34	
3	2,664	13.32	
4	1,870	9.35	
Total	20,000	100.00	

- Rounding does not work.
- The extra seat should go to the state with the
  - smallest population p_i
  - largest remainder $r_i = q_i - \lfloor q_i \rfloor$
  - largest relative remainder r_i / p_i

Hamilton's Method

Give to each state the whole number contained in its quota, and then assign remaining seats to states with the largest quota remainders.

State i	Population p_i	Quota q_i	Apportionment a_i
1	9,598	47.99	47 + 1 = 48
2	5,868	29.34	29 + 0 = 29
3	2,664	13.32	13 + 0 = 13
4	1,870	9.35	9 + 1 = 10
Total	20,000	100.00	100

Jefferson's Method

Choose an ideal district size. Give each state its whole number of seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State i	Population p_i	Districts p_i / 200	Apportionment a_i
1	9,598	47.99	47
2	5,868	29.34	29
3	2,664	13.32	13
4	1,870	9.35	9
Total	20,000		98

State i	Population p_i	Districts p_i / 195.7	Apportionment a_i
1	9,598	49.04	49
2	5,868	29.98	29
3	2,664	13.61	13
4	1,870	9.56	9
Total	20,000		100

Webster's Method

Choose an ideal district size. Give each state its **rounded** number of seats. If the house size is fixed, the ideal district size must be chosen so that the seats assigned matches the house size.

State i	Population p_i	Districts p_i / 200	Apportionment a_i
1	9,598	47.99	**48**
2	5,868	29.34	29
3	2,664	13.32	13
4	1,870	9.35	9
Total	20,000		**99**

State i	Population p_i	Districts p_i / **198**	Apportionment a_i
1	9,598	48.47	48
2	5,868	29.63	30
3	2,664	13.45	13
4	1,870	9.44	9
Total	20,000		100

Hill's Method

Choose the apportionment that minimizes the relative difference in average representation between pairs of states.

i	p_i	a_i	a_i
2	5,868	29	30
4	1,870	10	9
Pairwise Measure of Inequity		$\dfrac{\frac{10}{1870} - \frac{29}{5868}}{\frac{10}{1870}}$	$\dfrac{\frac{30}{5868} - \frac{9}{1870}}{\frac{30}{5868}}$
		= 0.0758	= **0.0586**

For our example, Hill's and Webster's methods yield the same apportionment. For some distributions of population, the two methods give different results.

Divisor Methods

Choose an appropriate district size λ. State i receives p_i / λ , **rounded** with respect to a divisor criterion, seats. OR

Choose an apportionment that minimizes a pairwise measure of inequity.

Method	Divisor	Inequity Measure
Jefferson	$a+1$	$a_i(p_j / p_i) - a_j$
Webster	$a+1/2$	$a_i / p_i - a_j / p_j$
Hill	$\sqrt{a(a+1)}$	$\dfrac{a_i / p}{a_j / p_j} - 1$
Dean	$\dfrac{a(a+1)}{a+1/2}$	$p_j / a_j - p_i / a_i$
Adams	a	$a_i - a_j(p_i / p_j)$

Does it Make a Real Difference?

For the 1990 Census

State	Quota	Hamilton	Webster	Hill
Massachusetts	10.552	11	11	10
Oklahoma	5.516	5	5	6
New Jersey	13.536	14	13	13
Mississippi	4.518	4	5	5

If Jefferson's method had been used, 16 states would have been apportioned different numbers of seats.

For the 2000 Census

Webster is the same as Hill. Hamilton takes a seat from California and gives it to Utah. Jefferson adds two seats to California among several other changes.

Does it Make a Real Difference?

"Since the world began there has been but one way of proportioning numbers, namely, by using a common divisor, by running the 'remainders' into decimals, by taking fractions above .5, and dropping those below .5; nor can there be any other method. This process is purely arithmetical . . . If a hundred men were being torn limb from limb, or a thousand babes were being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog."

Representative John A. Anderson of Kansas
Congressional Record 1882, 12:1179

What Method is Best?

“Since the world began there has been but one way of proportioning numbers, namely,

<insert your favorite method here>

nor can there be any other method. This process is purely arithmetical . . . If a hundred men were being torn limb from limb, or a thousand babes were being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog.”

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Which Method is Best?

- Method definitions are *ad hoc*.
- Huntington (1928) made the first systematic study of methods based upon measures of inequity.
- Balinski and Young (1982) use an axiomatic approach based upon desirable properties.
- More recent work includes Gonzalez and Lacourly (1992) and Petit and Terouanne (1990),

Fair Share

The number of seats assigned a state should be its quota rounded down or up.

State i	Population p_i	Quota q_i	Jefferson a_i
1	9,598	**47.99**	**49**
2	5,868	29.34	29
3	2,664	13.32	13
4	1,870	9.35	9
Total	20,000	100.00	100

Jefferson's method does not satisfy fair share.

No divisor method satisfies fair share.

Hamilton's method satisfies fair share.

House Monotonicity

No state loses a seat when the house size increases (populations unchanged).

State	100 seats		101 seats	
i	q_i	a_i	q_i	a_i
1	47.99	48	48.47	49
2	29.34	29	29.63	30
3	13.32	13	13.45	13
4	9.35	**10**	9.44	**9**
Total	100.00	100	101.00	101

Hamilton's method does not satisfy house monotonicity.

All divisor methods satisfy house monotonicity.

There are methods satisfying both fair share and house monotonicity.

Population Monotonicity

No state that increases its population should lose a seat to another state that decreases its population (house size unchanged).

State	First Census			Second Census		
i	p_i	q_i	a_i	p_i	q_i	a_i
1	9,598	47.99	48	9,550	47.99	48
2	**5,868**	29.34	**29**	**5,865**	29.47	**30**
3	2,664	13.32	13	2,610	13.12	13
4	**1,870**	9.35	**10**	**1,875**	9.42	**9**
Total	20,000	100.00	100	19,900	100.00	100

Hamilton's method does not satisfy population monotonicity.

All divisor methods satisfy population monotonicity.

There is **no** method satisfying both fair share and population monotonicity.

Partial Population Monotonicity

No state that increases its relative population should lose a seat to another state that decreases its relative population (house size unchanged).

State	First Census			Second Census		
i	p_i	q_i	a_i	p_i	q_i	a_i
1	9,598	47.99	48	9,550	47.99	48
2	5,868	**29.34**	**29**	5,865	**29.47**	**30**
3	2,664	13.32	13	2,610	13.12	13
4	1,870	**9.35**	**10**	1,875	**9.42**	**9**
Total	20,000	100.00	100	19,900	100.00	100

Hamilton's method satisfies partial population monotonicity.

Since population monotonicity implies partial population monotonicity, all divisor methods satisfy partial population monotonicity.

Near Fair Share

The transfer of a seat from one state to another does not simultaneously take both states closer to their quota.

State i	Population p_i	Quota q_i	First a_i	Second a_i
1	9,598	**47.99**	**47**	**48**
2	5,868	**29.34**	**30**	**29**
3	2,664	13.32	13	13
4	1,870	9.35	10	10
Total	20,000	100.00	100	100

Hamilton’s method satisfies near fair share.

Websters’s method is the unique method satisfying near fair share and population monotonicity.

Near fair share is independent of fair share.

Unbiased

The probability that state i is favored over state j equals the probability that state j is favored over state i. State i is favored over state j if

$$a_i / p_i > a_j / p_j$$

Quota	Jefferson	Webster	Hill	Dean	Adams
9.988	**11**	10	10	10	10
9.064	9	9	9	9	9
7.182	7	**8**	7	7	7
5.260	5	5	**6**	5	5
3.321	3	3	3	**4**	3
1.185	1	1	1	1	**2**

There is a clear ordering in the five traditional divisor methods from bias towards large states (Jefferson) and bias towards small states.

Under a variety of reasonable assumptions about the population probability distribution, Hamilton's method is unbiased and Webster's method is the unique unbiased and proportional divisor method.

Summary

Property	Hamilton	Webster	Hill	Jefferson
Fair Share	Yes	No	No	No
Near Fair Share	Yes	Yes	No	No
Unbiased	Yes	Yes	No	No
Population Monotone	No	Yes	Yes	Yes
Partial Population Monotone	Yes	Yes	Yes	Yes
House Monotone	No	Yes	Yes	Yes

Conclusion

Webster's or Hamilton's method would be an improvement upon Hill's method.