WHAT CONVINCES ABOVE-AVERAGE MATHEMATICS STUDENTS?

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Participants

• Eleven undergraduates who had received As and Bs in math courses

• Group 0: 4 students had no proof-based course (all 1\textsuperscript{st} year)

• Group 1: 3 students had 1 proof-based course (all 2\textsuperscript{nd} year)

• Group 2\textsuperscript{+}: 4 students had at least 2 proof-based courses (2\textsuperscript{nd}, 3\textsuperscript{rd}, and two 4\textsuperscript{th} year)
Data Collection

For 40 minutes, students examined 7 conjectures, stated whether each conjecture was true or false, and provided proofs.

In a 20 minute interview, students were asked for each conjecture the following:

- How certain are you that the conjecture is true or false?
- How convincing is your proof to you?
- How convincing would your proof be to a peer?
- How convincing would your proof be to a mathematician?
Conjectures

1. The sum of the three interior angles of any triangle is 180 degrees.

2. If no angle of a quadrilateral is obtuse, then the quadrilateral is a rectangle.

3. If \((a + b)^2\) is even, then \(a\) and \(b\) are even.

4. The product of two negative real numbers is always a positive real number.

5. A polynomial of degree 3 must have at least one real root.

6. If \(A\) is a subset of \(C\) and \(B\) is a subset of \(C\), then the union of \(A\) and \(B\) is a subset of \(C\).

7. If an operation \(*\) is commutative, then \(*\) is associative.
# How Convincing Did Students Believe Their Proofs to Be?

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<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>20</td>
<td>Very convincing to all</td>
</tr>
<tr>
<td>9</td>
<td>Very convincing to self but not to a mathematician</td>
</tr>
<tr>
<td>12</td>
<td>Only somewhat convincing to self</td>
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<tr>
<td>36</td>
<td>Unconvincing</td>
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# How Many Proofs Did the Researchers Find Convincing (With Caveats)?

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>7</td>
<td>Deductive proofs (2 considered only somewhat convincing by students)</td>
</tr>
<tr>
<td>6</td>
<td>Counter-examples (1 considered unconvincing by student)</td>
</tr>
<tr>
<td>3</td>
<td>Counter-arguments</td>
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Mathematical Correctness versus Student Perception

<table>
<thead>
<tr>
<th></th>
<th>All</th>
<th>Self</th>
<th>Some</th>
<th>None</th>
<th>Total</th>
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<tbody>
<tr>
<td>Correct Proof</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Minor Mistakes</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Substantial Progress</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>Minimal Progress</td>
<td>3</td>
<td>7</td>
<td>9</td>
<td>35</td>
<td>54</td>
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<tr>
<td>Total</td>
<td>20</td>
<td>9</td>
<td>12</td>
<td>36</td>
<td>77</td>
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</tbody>
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Students' Proof Schemes
Guershon Harel and Larry Sowder
CBMS issues in Mathematics Education, 1998
Transformational Proof Scheme Expressions

- 6 students provided 7 proofs of true conjectures that were judged to be fully deductive with at most minor mistakes (2, 1, 3*).
  - 2 were considered flawless.
  - 5 expressed concern about the truth of or acceptability of using certain results in their proofs.

- 6 counter-examples given for false conjectures (2, 2, 2)
Axiomatic Proof Scheme Expressions

• One group 2\(^{+}\) student understood the Non-Euclidean counter-example to Conjecture 1

• One group 1 student understood the Non-Euclidean counter-example to Conjecture 1 and used multiple nonstandard definitions when considering Conjecture 6

  No one made substantial progress on Conjecture 7 other than noting that the conjecture would be true if only the basic four operations (+, −, ×, ÷) were considered.
Counter-Arguments

• A group 0 student used two counter-arguments instead of counter-examples.

• A group 2+ student provided a substantial counter-argument instead of a counter-example to Conjecture 3.

• Another group 0 student used a counter-argument in addition to a counter-example because she was unsure what interviewer was looking for.
Perceptual Proof Scheme Expressions

• only special cases examined
• expression of conjecture in informal language considered a proof
Inductive Proof Scheme Expressions

• One group 0 student used examples to justify
  • 1\textsuperscript{st} and 2\textsuperscript{nd} year students used examples to illustrate, but not justify, some conjectures
Ritual Proof Scheme Expressions

• 5 students found at least one proof would be less convincing to a mathematician than to themselves because of the form (3, 1, 1)

• 2 students found at least one proof very convincing to themselves that involved serious framework errors (1, 0, 1)
  - 5 students found at least one proof would be less convincing to others than to themselves because of insufficient clarity (1, 2, 2)
  - 4 students found at least one proof would be less convincing to a mathematician than to themselves because of the results used (0, 2, 2)
Why A Proof Is Less Convincing To Mathematicians

• I haven’t done proof in a long time, so I didn’t quite remember the format. I just kind of wrote down what I thought.

• It needs to be a little more technical.

• I don’t really know quite what’s acceptable in writing a proof.

• I’ve forgotten how to go through and like the exact mathematical whatever I was supposed to do.

• People are mostly looking like for a ‘proof’ proof, and this is more like a ‘feeling’ proof . . . . A ‘feeling’ proof is like you just go by what you know, but there’s no like concrete like mathematical terms.

• I guess because I think it’s too short.
Symbolic Proof Scheme Expressions

- 8 instances of nonstandard or incorrect symbolic notation that was considered at least somewhat convincing (4, 3, 1)
  - One group 0 student avoided symbolic notation completely
Authoritarian Proof Scheme Expressions

• 8 students made explicit appeals to authority, including purported rules, when discussing Conjectures 1 and 4 (4, 2, 2)

• 10 students were very certain of the truth of some conjecture without being able to exhibit any proof even somewhat convincing to themselves (3, 3, 4)

• 2 students were more sure of the truth of Conjecture 1 than Conjecture 2 even though no proof was given for Conjecture 1 and a “very convincing” proof was given for Conjecture 2 (0, 0, 2)
Explicit Appeals To Authority

• I said true because I remember from like sixth grade, you say you add up all the angles to $180^\circ$.

• Because this is something that I have just known. That I’ve been told going all the way back to junior high that this is just something that I’ve just accepted. I haven’t really thought much about trying to prove it.

• I know, because that’s in a textbook.

• I don’t think I’ve ever seen a proof of that. It kind of goes along with the ah, when they tell you, you agree.

• So if we define it as that, then what is there to prove from that?

• I knew if all of them were true or false, I just didn’t know how to prove them.