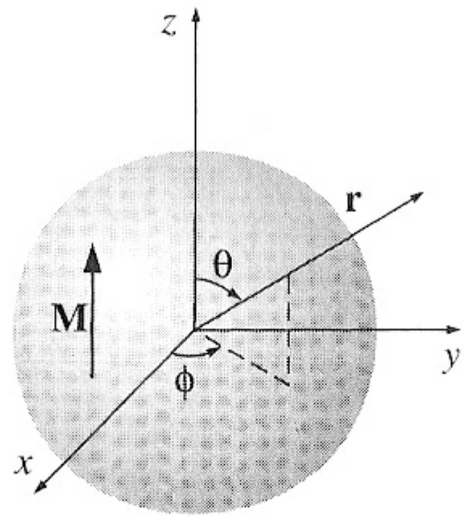
Problem 6.15 - Uniformly magnetized sphere

Find the magnetic field inside and outside a uniformly magnetized sphere.

It was pointed out that the curl of the auxiliary field is zero, so there's a potential such that $\vec{\nabla} W = \vec{\mathbf{H}}$. And...

- $\nabla^2 W = 0$: That is, $\vec{\mathbf{W}}$ satisfies Laplace's equation.
- W is continuous across the boundary of the sphere at $r = R$.



The most general solution to Laplace's equation for azimuthally symmetric conditions in spherical coordinates can be written in terms of the Legendre polynomials.

$$W = \sum_{l=0}^{\infty} \left(A_l r^l + B_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta).$$

Where the first couple of Legendre polynomials are given in the table.

l	$P_l(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

We'll use boundary conditions to simplify the general solution:

Outside the sphere, the potential $W_{\text{out}}(r, \theta)$ should not diverge as $r \to \infty$. This allows you to set either all the A_ls or all the B_ls to zero. Write down the form of the solution outside the sphere that has only finite terms as $r \to \infty$.

$W_{\text{out}} =$

Inside the sphere, $W_{\text{in}}(r, \theta)$ needs to stay finite as $r \to 0$. Starting from the most general solution, write down a simpler version, W_{in}, that obeys this constraint:

$W_{\text{in}} =$

At the surface of the sphere ($r = R$):

$$H_\perp^{\text{out}} - H_\perp^{\text{in}} = -(M_\perp^{\text{out}} - M_\perp^{\text{in}}).$$

(1)

Evaluate the following derivatives:

- $H_\perp^{\text{out}}\Big|^{r=R} = -\frac{\partial}{\partial r} W_{\text{out}}\Big|^{r=R} =$
- $H_\perp^{\text{in}}\Big|^{r=R} = -\frac{\partial}{\partial r} W_{\text{in}}\Big|^{r=R} =$
- $M_\perp^{\text{outside}} = ?.$
- $M_\perp^{\text{inside}} = \vec{\mathbf{M}}_r = \left(M_0\hat{\mathbf{z}}\right)_r = M_0 \cos\theta = M_0 P_n(\cos\theta).$
  Which Legendre polynomial is this? That is...$n = ?$

Write out (1) in terms of the four quantities above:

{eq *1}

Since the Legendre polynomials are orthogonal, only the the terms where $l = n$ are other than zero. Write out {1*} for the particular integer for which this is true:

{eq *2}

Now, use the continuity of W across the surface of the sphere:

$$W_{\text{in}}(r = R) = W_{\text{out}}(r = R)$$

Substitute your expressions for the potential inside and outside, and solve this for the coefficients B_n.

{eq *3}

Substitute this B_n into {2*} to solve for A_n:

{*4}

OK, so now you should be in good position to write down a fairly simple expression for W_{in}. Do that, and then figure out $\vec{\mathbf{H}}_{\text{in}} = -\vec{\nabla} W_{\text{in}}$.

{eq *5}

Finally, substitute your $\vec{\mathbf{H}}$ into $\vec{\mathbf{B}} = \mu_0(\vec{\mathbf{H}} + \vec{\mathbf{M}})$ to calculate the $\vec{\mathbf{B}}$-field.

{eq *6}

Which of these pictures is the $\vec{\mathbf{B}}$-field and which is the $\vec{\mathbf{H}}$-field?